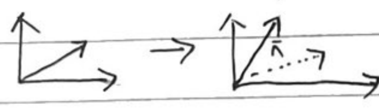
15th May 2022

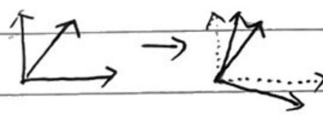
Special Relativity Lecture 1

Why change in coordinates are needed?

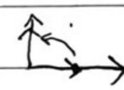
Difference between active and passive transformations of coordinate systems?

"active": we consider object to be rotated in a fixed coordinate system i.e. physically moving the object. 

To rotate back, the inverse matrix of the rotation matrix is multiplied to both sides. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

"Passive" The frame of the coordinate is rotated instead. 

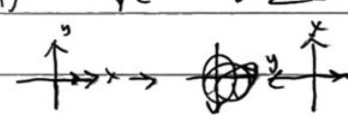
For a point $(1, 0)$

active rotation by $90°$ $\rightarrow$ $(x' \; y') = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (anticlockwise).

Passive rotation by $90°$ $\rightarrow$ $(x' y') = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ~~clockwise~~

both can be used if there is a perfect symmetry in all directions of space. 

What are invariants and covariants?

Invariant: another word for scalar (doesn't change under rotation)

Covariants: The vectors on both sides of the equation change together in the same way, like $\vec{F} = m\vec{a}$, upon rotation.

- All physical laws must be able to be described in covariant equations.

- Active rotation by ϕ is mathematically equivalent to passive rotation by $-\phi$

How are inertial frames used in relativity ?

In relativity, the transformation is not due to angle change, but a change in velocity. All physical laws still need to be valid after transformation.

Special relativity is the case where frame is moving only with fixed velocity.

Inertial pretty much means fixed velocity.

$x = ut \quad x' = u't \quad x' = (u-v)t$

$x' = x - vt + c$

What is the Galilean transformation

It is the application of the inertial frame in classical physics

It turns out that the Galilean transformations are not completely correct

e.g. A train travelling at ~~too~~ $50ms^{-1}$ and you throw something at lightspeed.

Special Relativity

Lecture 2

- Why was the aether proposed and how did we conclude it did not exist?

It turns out that Maxwell's equations are not covariant under the classical Galilean transformation equations.

For sound waves we can tell what frame we are in because the speed changes to a moving frame. (wind)

The aether was postulated to be the medium for light waves. The theory ~~only~~ assumes the equation for the speed of light, hence the Maxwell equations, only hold in one frame. This theory was tested by comparing the measured speeds of light. (Fringes in the Michelson-Morley experi)

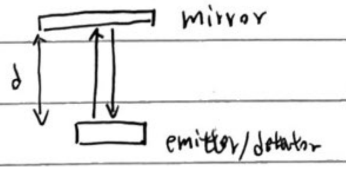
- What are the postulates of relativity?

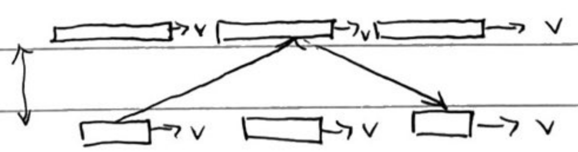
1. The laws of physics are the same in all inertial frames
2. The speed of light in vacuum is independent of the ~~of the~~ speed of the lightsource and has the same value for all inertial observers.

- observer means a system that records time stamped images ~~of~~ in all space at all times.

- What is the smallest possible value of the relativistic gamma parameter?

The light clock:





At rest: Time taken for one trip is $T = \frac{2d}{c}$

Moving: let period be T', time for light to get to mirror is $\frac{T'}{2}$

The system moves $v\frac{T'}{2}$, light goes $c\frac{T'}{2}$, using ~~Pa~~ Pythagoras

$$T' = \frac{T}{\sqrt{1-v^2/c^2}} = \frac{T}{\sqrt{1-\beta^2}} = \gamma T$$

where $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ $\qquad \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1$

γ and $\gamma\beta$ are dependent on β; when $\beta = 0$ $\gamma = 1$ (smallest)

How is the period, T', of a clock observed in a moving frame S' related to it's period, T, in it's rest ~~...~~ frame, S? Does this effect depend on the type of clock?

$$T' = \gamma T \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

? This applies to both the clock (active transformation) and to the observer (passive transformation)

√? How do we reconcile this with the fact that an observer in the frame S' would observe her own clock to be running slower than that in the frame S, which from her perspective would be the one moving i.e. she would observe $T' < T$?

Twin's paradox, time dilation works both ways, all inertial frames are equivalent. Time depends not just on relative speed, but also position.

What is the proper time?

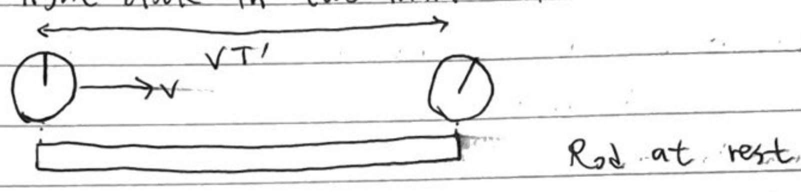
When $\gamma = 1$ $T' = T$ is the proper time with symbol τ.

Special Relativity

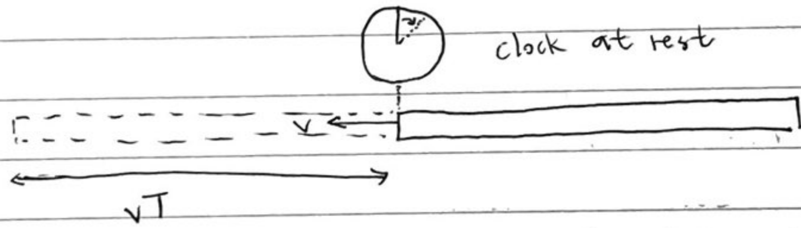
Lecture 3

- Why do we get length contraction?

Consider a light clock in two inertial frames



Clock is moving with speed v so that the length of the rod is the ~~hour~~ distance travelled by the clock in T'. So the rod has length vT'

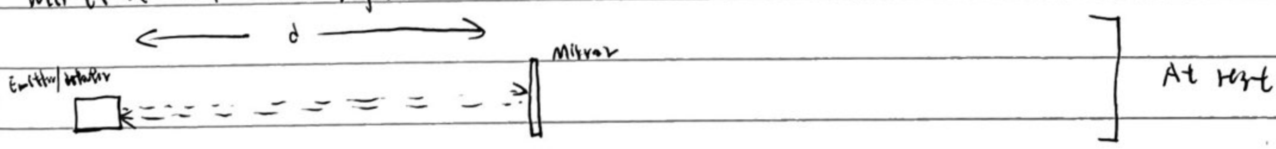


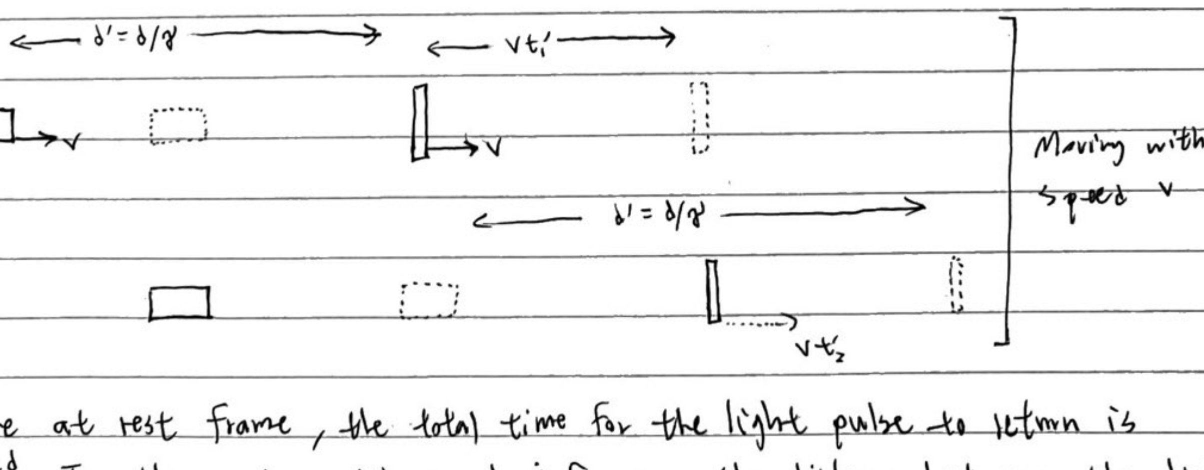
Now consider this to an observer in the clock rest frame, the length of the moving wooden rod must be $l' = vT = \frac{vT'}{\gamma}$ $\quad l' = \frac{l}{\gamma}$

l is called 'proper length'.

- Are two simultaneous occurances that occur at different x positions in some frame also simultaneous in a frame moving along x?

We now look at the light clock in a frame moving parallel to the motion of the light.





In the at rest frame, the total time for the light pulse to return is $T = \frac{2d}{c}$. In the moving with speed v frame, the distance between the light source and the mirror will be shortened to $\frac{d}{\gamma}$. $\left(d' = \frac{d}{\gamma}\right)$

The time, t_1', for the light to reach the mirror is given by

$$ct_1' = \frac{d}{\gamma} + vt_1'$$

Where ct_1' is the total distance travelled by the light; $\frac{d}{\gamma}$ is d' the length after Lorentz contraction; and vt_1' the distance moved by the mirror during this time. So

$$t_1' = \frac{d}{\gamma(c-v)} = \frac{d}{\gamma(1-\beta)c}$$

Since the source/detector are also moving, the time t_2' for the light to return from the mirror to the source is

$$ct_2' = \frac{d}{\gamma} - vt_2', \text{ so}$$

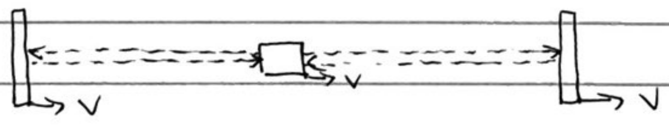
~~Since~~ $$t_2' = \frac{d}{\gamma(c+v)} = \frac{d}{\gamma(1+\beta)c}$$

The total time for the light to return, i.e. the period, is

$$T' = t_1' + t_2' = \frac{d(1+\beta) + d(1-\beta)}{\gamma(1-\beta)(1+\beta)c} = \frac{2d}{\gamma(1-\beta^2)c} = \frac{2d\gamma}{c} = \gamma T$$

Hence the time dilation factor doesn't depend on which way the clock is oriented.

Now we consider a double light clock moving at v with the same distance.



This is identical to the above ~~situ~~ situation where the time to reach the front and back mirrors are $\frac{d}{\gamma(1-\beta)c}$ and $\frac{d}{\gamma(1+\beta)c}$

and the time taken to bounce back are $\frac{d}{\gamma(1+\beta)c}$ and $\frac{d}{\gamma(1-\beta)c}$

Hence the lights arrive at the center at the same time, though they don't hit the mirror at the same time.

☆ The general principle is that for two ~~simul~~ simultaneous occurrences which are not ~~at~~ at the same x position, e.g. lights hitting two mirrors in rest frame, they won't be simultaneous in any other frame moving along x.

Hence, to be simultaneous in any frame moving in any direction, the two occurrences must be at exactly the same position (e.g. hitting the emitter/detector)

? How does the relativity of simultaniety help understand e.g. length contraction?

Two simultaneous occurances which are not in the same position are no longer simultaneous if viewed in any other inertial frame.

For measurement ~~for~~ of lengths, in rest frame two ends are simultaneously measured as the proper length. In moving frame, measurement is not simultaneous as ~~the~~ time taken to hit the mirrors are different, so different ~~l~~ l' since $l' = \frac{l}{\gamma}$ and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

25th May 2022

Spacial Relativity Lecture 4 Lorentz transformations

- The concept of an event
  - If just considering observers moving along the x axis, then we actually only have to consider values of t and x. An event is something happening at a specific x at a particular time t.
  - A true event occurs at an infinitessimally small point in space at an instant in time, but we usually use approximations.
  - Proper time is the time between two events in an inertial frame in which the events have the same position
  - Proper length is the distance between two events in the frame in which the events have the same time and in which the object they mark the ends of is stationary.
- The Lorentz transformations
  - Given an event at t, x, y, z, then a passive Lorentz transformation from an initial frame to the frame of an observer moving at speed v along the $+x$ axis can be written as

$$t' = \gamma(t - \frac{vx}{c^2}) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z$$

where t' and x' tell you the coordinates of the event in the different frame. The change from t to t' depends on speed (v) and position (x). $t=0, x=0$ always transform to $t'=0, x'=0$. The transform can also be rewritten as: $t' = \gamma(t - \frac{\beta x}{c}) \quad x' = \gamma(x - \beta ct)$

we can multiply the first equation by c to get

$$ct' = \gamma(ct - \beta x) \qquad x' = \gamma(x - \beta ct)$$

which shows symmetry between ct and x, writing them in matrix equation:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\boxed{\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}}$$

To inverse transform, we find the inverse matrix

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

- Small speed approximation
  - we also need to make sure that an Lorentz transformation corresponds to the classical Galilean transform. If β is small, then $\gamma \approx 1$ so the Lorentz transform is approximately:

$$ct' \approx ct - \beta x \qquad x' \approx x - \beta ct \qquad \beta = \frac{v}{c}$$

$$t' \approx t - \frac{vx}{c^2} \qquad x' \approx x - vt$$

$$\approx t$$

- Consistency of the speed of light
  - Say a light pulse is generated from the origin $x=0$ at $t=0$ and travels along the x axis so after time T will be at $x=cT$. Take $t=0$ as one event and $t=T$ for another. The first event in ~~one~~ another frame also has the coordinates zero. For the second event,

$$cT' = \gamma(cT - \beta cT) \qquad x' = \gamma(cT - \beta cT)$$

The speed in the boosted frame is $u' = \frac{\Delta x'}{\Delta T'} = \frac{x'}{T'}$,

$$u' = \frac{x'}{T'} = \frac{\gamma(cT - \beta cT)}{\gamma(cT - \beta cT)/c} = c$$

Lorentz transformation preserves the speed of light.

- Velocity transformation

For an object moving at a speed $u<c$, it goes from the origin to uT in time T. Applying the Lorentz transformation:

$$cT' = \gamma(cT - \beta uT) \qquad x' = \gamma(uT - \beta cT)$$

The speed in the boosted frame is $u' = x'/T'$:

$$u' = \frac{x'}{T'} = \frac{\gamma(uT - \beta cT)}{\gamma(cT - \beta uT)/c} = \frac{u - \beta c}{1 - \beta u/c} = \frac{u - v}{1 - uv/c^2}$$

Although it is calculated from the Lorentz transformation formula, it's not a Lorentz transformation; velocity doesn't change in the same way as position.

- If initial velocity $u=0$ then $u' = -v$ i.e. boosting to a frame going at v along the x axis makes an object initially at rest appear with $-v$

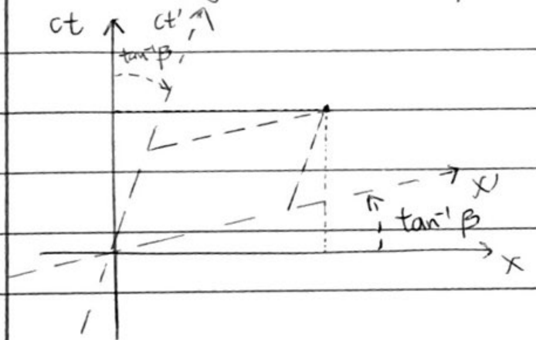
~~18th~~ 26 May 2022

Special Relativity Lecture 5 Space-time diagrams and world lines

- Space-time diagrams
  - An event is a single point in such a diagram
  - Any event on the x' axis has $t' = 0$ by definition

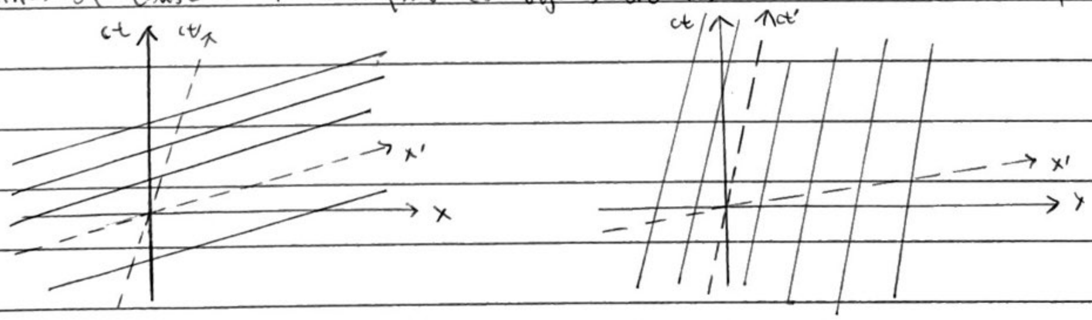
    $ct' = \gamma(ct - \beta x) = 0$ so $ct = \beta x$

    Similarly $x' = \gamma(x - \beta ct) = 0$ $ct = \frac{x}{\beta}$



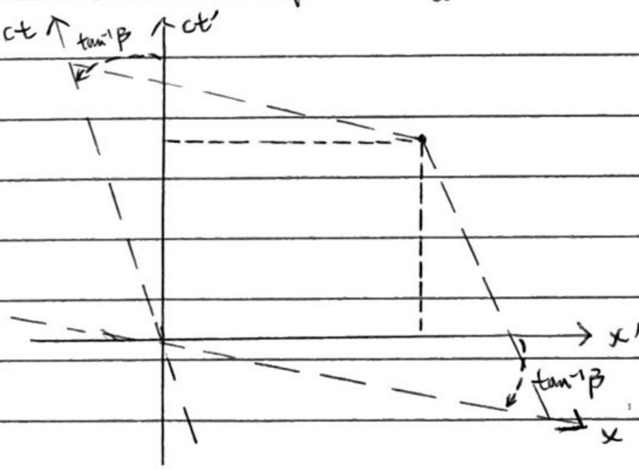
Passive Lorentz transformation by β of the coordinate system with a given space-time event

- Any line of constant t' is referred to as a 'line of simultaneity' because all events on that line are simultaneous in the primed (or unprimed) frame.
- Lines of constant x' correspond to objects at rest in the transformed frame



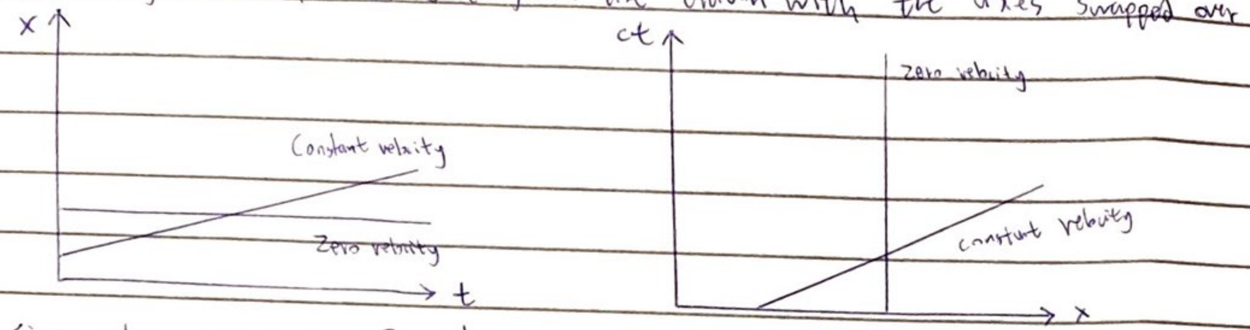
Left: Lines of constant t' (simultaneity) Right: lines of constant x' (rest)

- The observer always consider their axes to be perpendicular. From their perspective, the original frame, which is equivalent to an inverse L.T:



- World lines

· In relativity, the space-time diagrams are drawn with the axes swapped over

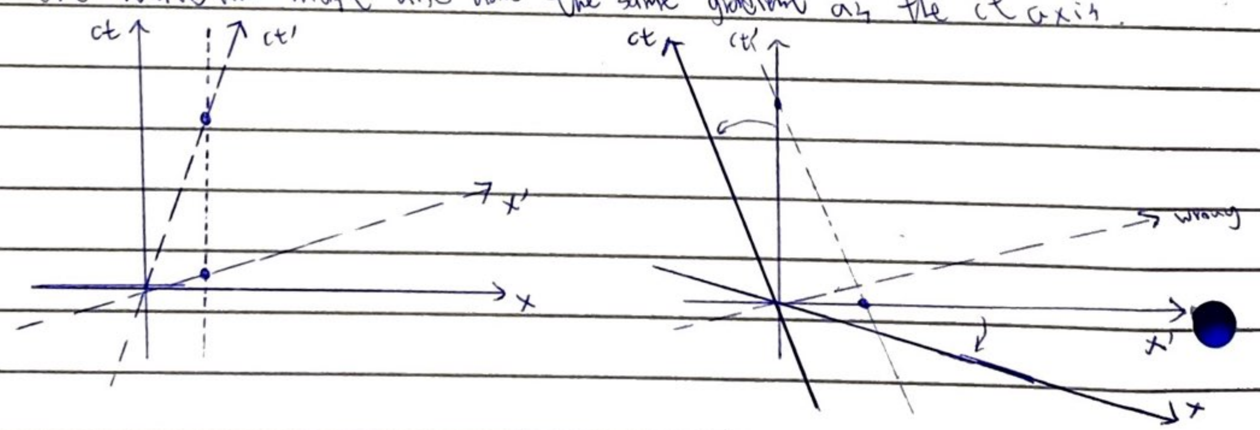


since the gradient is $\frac{c}{v} = \frac{1}{\beta}$

we are limited to $|u| \leq c$ hence $\frac{1}{|\beta|} \geq 1$. The magnitude of the gradient must be always at least 1. Light always travel at gradient = 1

- Transforming World lines

· An object in ~~the~~ a frame at rest has a world line parallel to the ct axis in that frame. Under Lorentz transformation, in the new frame, the ct axis in that frame will be tilted, hence the world line must also have the same gradient as the ct axis.



Note that simple geometry cannot be used to compare distance between inertial frames since the axes are both stretched and tilted. Always work in terms of events to be safe.

Special Relativity Lecture 6 Four-vectors and causality

- Four-vectors
  - have the form $(ct, \vec{r})$ or (ct, x, y, z)
  - A four-vector is what we mean by an event
  - Similar to the length of a three-vector $|\vec{v}|$ not changing after rotation, the length squared of a four-vector, S^2, is an invariant (Lorentz invariant)
  - $S^2 = (ct)^2 - |\vec{r}|^2$ S^2 can be positive, negative, or zero
- Event Separation
  - Similar to three vectors, the separation between two events is defined to be the length squared of the four-vector resulting from subtracting the four-vectors of the two events.

    $(c\Delta t, \Delta\vec{r}) = (ct_2 - ct_1, \vec{r}_2 - \vec{r}_1)$

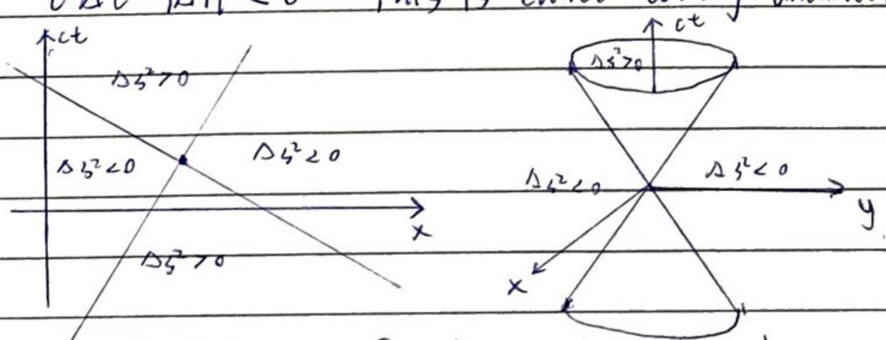
    $\Delta S^2 = c^2\Delta t^2 - |\Delta\vec{r}|^2$

    It is also a Lorentz invariant.
- Causality
  - For one event to cause another, we require the distance to be less than or equal to the distance that light can go in that time.

    $|\Delta\vec{r}| \leq c\Delta t \rightarrow |\Delta\vec{r}|^2 \leq c^2\Delta t^2 \rightarrow c^2\Delta t^2 - |\Delta\vec{r}|^2 \geqslant 0$

    i.e. the invariant interval must not be negative.
  - One event cannot cause another if the invariant interval is negative

    $c^2\Delta t - |\Delta\vec{r}|^2 < 0$ This is called "causally unconnected"



Left: the light cone for the event marked by a circle and corresponding separations in ct, x. Right: 3D visualization of light-cone in ct, x, y, at origin.

1. For $\Delta s^2 > 0$, the time order of two events is the same in all frames and the first can affect the second. These events are called "time-like" separated because there is always one frame in which they have no difference in position, but are separated in time. The separation is therefore $\Delta s^2 = c^2 \tau^2$ where τ is the proper time between the events. The space order is different in different frames.
2. For $\Delta s^2 < 0$, the time order of two events is different in different frames and so they cannot logically affect each other. The space order is always preserved. These events are called "space-like" because there is always one frame in which they have no difference in time but are separated in space.
3. $\Delta s^2 = 0$ is a special case. The events are only connected by a light-speed signal and lie exactly on the diagonals of the light cone. They retain their time and space order for any boost. In principle the first can affect the second; hence they are causally connected, as for ~~time~~ time-like separated events. However, because c is the same in all frames, they always lie on the diagonals and there is no frame in which they have either a zero time or a zero space difference. These are called 'light-like' separated, for obvious reasons.

- Tachyons
  - a hypothetical particle going faster than the speed of light
  - Going backward in time in some frames
  - Assumed cannot exist

7th June 2022

Special Relativity Lecture 7

- Energy and momentum

- The energy E and momentum $\vec{p}$ of any object form a four-vector $(E, \vec{p}c)$ called the 'four momentum'. Since all four-vectors change in the same way:

$$\begin{pmatrix} E' \\ p'_x c \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x c \end{pmatrix}$$

- In the rest frame $\vec{p} = 0$ but E cannot be 0, so let the rest frame energy be E_0
- Consider the passive transformation of the object at rest $(u=0)$ in the x direction by v, so that object will be moving at $u' = -v$ in that frame. We can define $\beta'_u = \frac{u'}{c} = \frac{-v}{c} = -\beta$ and $\gamma'_u = \frac{1}{\sqrt{1-\beta'^2_u}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma$

We got:

$$E' = \gamma E_0 = \gamma'_u E_0 \quad \text{and} \quad p'c = -\gamma\beta E_0 = \gamma'_u \beta'_u E_0$$

- E_0 can be found: For small β'_u, $\gamma'_u \approx 1$ so $p' = \frac{\gamma'_u \beta'_u E_0}{c} \approx \frac{u' E_0}{c^2}$
We know that a particle moving at non-relativistic speed u in the x direction, momentum $p = mu$ so $m = \frac{E_0}{c^2}$ i.e. $E_0 = mc^2$
This energy is called the rest energy.
- Now we can obtain the exact formulae for energy and momentum in any other frame: $E' = \gamma'_u mc^2$ $\quad p'c = \gamma'_u \beta'_u mc^2 \rightarrow p' = \gamma'_u mu'$
- These equations only involve quantities all measured in the same frame, so we drop the primes from now on.

Kinetic energy $K = E - mc^2 = (\gamma_u - 1)mc^2$

momentum $\vec{p} = \gamma_u m \vec{u}$

$\vec{u} = \frac{\gamma_u mc^2 \vec{u}}{\gamma_u mc^2} = \frac{\vec{p}c^2}{E}$ and $\vec{\beta}_u = \frac{\vec{u}}{c} = \frac{\vec{p}c}{E}$

$\gamma_u = \frac{\gamma_u mc^2}{mc^2} = \frac{E}{mc^2}$ and $\gamma_u \vec{\beta}_u = \frac{\vec{p}c}{E}\frac{E}{mc^2} = \frac{\vec{p}}{mc}$

- For the expression $E = \gamma_u mc^2$, $\gamma_u \rightarrow \infty$ as $u \rightarrow c$ hence $E \rightarrow \infty$ too.

- Small speed approximation

We can get a better approximation for $\gamma_u = (1-\beta_u^2)^{-\frac{1}{2}}$.

$\gamma_u \approx 1 + (-\frac{1}{2})(-\beta_u^2) \approx 1 + \frac{1}{2}\beta_u^2$

So $E = \gamma_u mc^2 \approx mc^2 + \frac{1}{2}mc^2\beta_u^2 \approx mc^2 + \frac{1}{2}mu^2$

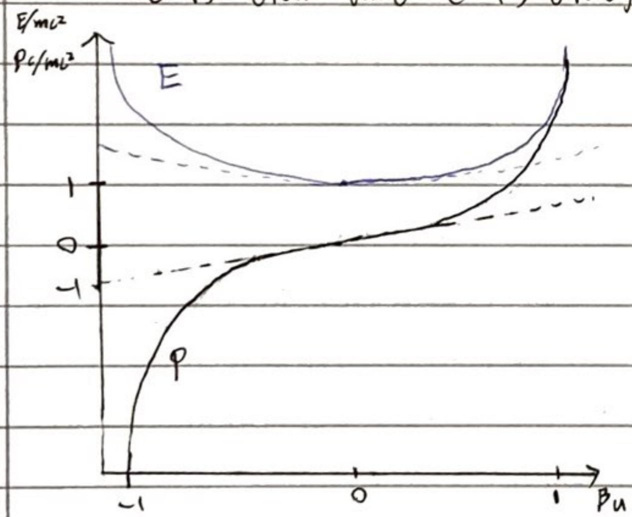
- ~~Energy~~ -

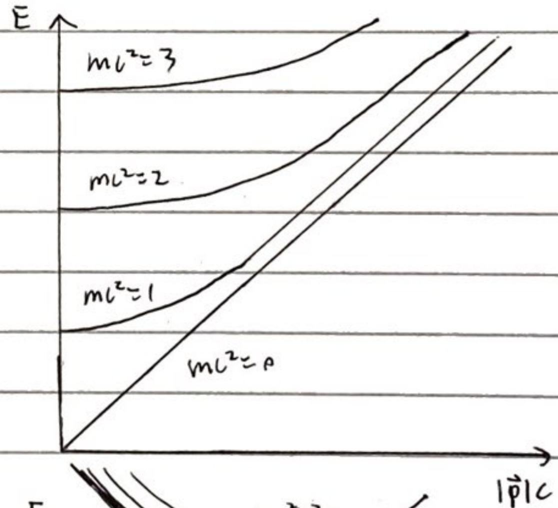
- Energy-momentum invariant
  - The length-squared for every four-vector must be an invariant

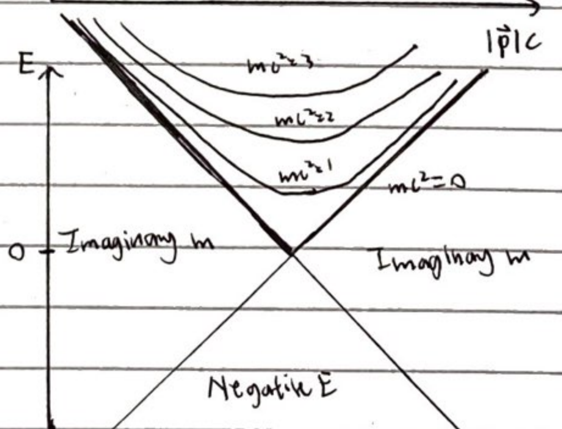
    $E^2 - (pc)^2 = \gamma_u^2 m^2 c^4 - \gamma_u^2 \beta_u^2 m^2 c^4 = \gamma_u^2 (1-\beta_u^2) m^2 c^4 = (mc^2)^2$

    This is the rest energy squared. A mass is a real value, so $(mc^2)^2$ is always positive. The above equation is often rearranged to give

    $E^2 = (pc)^2 + (mc^2)^2 = p^2c^2 + m^2c^4$
  - It is clear that E is always greater than pc and mc^2







- Velocity transformation
  - under Lorentz transformation

    $E' = \gamma(E - \beta pc)$, $p'c = \gamma(pc - \beta E)$

    $u' = \frac{p'c^2}{E'} = \frac{\gamma(pc^2 - \beta cE)}{\gamma(E - \beta pc)} = \frac{pc^2 - vE}{E - vp}$

    $= \frac{pc^2/E - v}{1 - vp/E} = \frac{u - v}{1 - \frac{vu}{c^2}}$

    This is the same as the one we got previously.
- Photons
  - $m=0$ $|\vec{u}| = c$. $\vec{\beta}_u = \frac{\vec{p}c}{E}$, $|\vec{\beta}_u| = 1$ so $E = |\vec{p}|c$
  - For all photons, $m=0$ $|\vec{u}| = c$, so do they have the same E and p?

    No, they also vary in frequency or equivalently wavelength.

    $E = hf$, $|\vec{p}| = \frac{h}{\lambda}$ In Quantum Mechanics

    Sub into $E = |\vec{p}|c \rightarrow hf = \frac{hc}{\lambda} \rightarrow f\lambda = c$

Special Relativity Lecture 8

- Units
  - In experimental tests of four-momentum in special relativity energies will be measured in eV. pc also has dimensions of energy so can be given the same units. So momentum can be written in units of eV/c. Similarly mc^2 can be given in $eV \rightarrow m$ can be given in eV/c^2
- Implications of the rest mass energy
  - Energy implies mass: Classically the mass of a box is simply $\Sigma_i m_i$. However $m = \frac{E_0}{c^2} \rightarrow$ the mass is the total energy of everything divided by c^2. Where energy includes rest mass, kinetic, potential, thermal, binding, etc.
    But in classical physics its approximately a change of 10^{-8} of the mass.
  - Mass implies energy: mass decreases will imply emission of energy
  - The electron is a fundamental particle with no internal structure, there is no way to put energy into an electron at rest. Hence fundamental particles have fixed values for mass.
- Mass conservation
  - Energy and momentum are conserved in all processes for an isolated system
  - The total mass m_T, is also conserved and is defined:
    $$E_T^2 = p_T^2 c^2 + m_T^2 c^4 \quad \text{so} \quad m_T = \frac{\sqrt{E_T^2 - p_T^2 c^2}}{c^2}$$
  - Note that m_T is not defined as $\Sigma_i m_i$ i.e. the sum of particle masses.
- Particle decay
  - Consider a particle decay in the decay particle rest frame. The initial energy is $E = m_0 c^2$ and initial momentum $\vec{p} = 0$. After the decay, energy conservation requires $E_1 + E_2 = m_0 c^2$. The 2 daughter particles must be back to back to conserve momentum i.e. $\vec{p}_2 = -\vec{p}_1 \rightarrow \vec{p}_2^{\,2} = \vec{p}_1^{\,2}$
    $$p_1^2 c^2 = E_1^2 - m_1^2 c^4 \qquad p_2^2 c^2 = E_2^2 - m_2^2 c^4$$
    momentum conservation requires $E_1^2 - m_1^2 c^4 = E_2^2 - m_2^2 c^4$
    $$E_1 = \frac{m_0^2 c^4 + m_1^2 c^4 - m_2^2 c^4}{2 m_0 c^2} \qquad E_2 = \frac{m_0^2 c^4 + m_2^2 c^4 - m_1^2 c^4}{2 m_0 c^2}$$
    which is symmetric under interchange of $1 \leftrightarrow 2$
  - The maximum amount of usable liberated energy in the rest frame is $K_1 + K_2 = (m_0 - m_1 - m_2)c^2$

Special Relativity Lecture 9

- Constant, invariant, and conserved quantities
  - A constant quantity has a fixed value. Examples are c, h, m_e, etc.
  - An invariant is a quantity that is unchanged under a transformation. However it can change with time within an inertial frame. The four-vector dot product is also a Lorentz invariant and the length-squared is just the dot-product of the four-vector with itself.
  - A conserved quantity doesn't change with time for a given physical system. It can have a different value in another frame but it's value in that frame should also be conserved. e.g. total energy E_T and total momentum $\vec{P}_T$ of an isolated system. They change under Lorentz transformation but in each frame they don't change with time.
  - Some quantities satisfy more than one of these e.g. c is constant, invariant, and conserved. m_T is both conserved and invariant.
- Center-of-mass
  - Consider two particles bouncing off each other, we can define the center of mass (CM) frame to be the one where the total momentum is zero. The total energy in this frame, often labelled E_{cm}, is $E_{cm} = E_T = m_T c^2$ even if there is no particle of mass m_T. Hence E_{cm} is actually invariant and conserved as it's directly related to m_T
  - The CM energy determines what reactions can happen. $E_T^2 = p_T^2 c^2 + m_T^2 c^4$ and m_T are the same in all frames. Since the momentum is non-zero in any other frame than the CM frame, E_T is higher in those frames. This higher incoming energy is used to provide the KE for the whole system hence is not available for the reaction, so doesn't change what reactions can happen.
- Particle reactions
  - Consider a photon being deflected by colliding with an electron in the CM frame $\gamma + e \rightarrow \gamma + e$ hence initial momenta $\vec{P}_{\gamma i} = -\vec{P}_{ei}$ and final momenta $\vec{P}_{\gamma f} = -\vec{P}_{ef}$ balance due to overall conservation. Energy conservation requires $E_T = E_{\gamma i} + E_{ei} = E_{\gamma f} + E_{ef}$. $\rightarrow$ they bounce off at some different angle and the final four momentum can be Lorentz transformed to see how this

appears in other frames. This reaction is known as compton scattering.

- We pretend there is a particle of mass m_T and this pretend particle then decays to give the out-going photon and electron. The pretend particle mass is now $m_0 = \frac{E_{cm}}{c^2}$, so:

$$E_1 = \frac{E_{cm}^2 + m_1^2c^4 - m_2^2c^4}{2E_{cm}} \quad \text{and} \quad E_2 = \frac{E_{cm}^2 + m_2^2c^4 - m_1^2c^4}{2E_{cm}}$$

- There are more complicated cases: $e^+ + e^- \rightarrow \gamma + \gamma$
  In the cm frame $E_{e^+} = E_{e^-} = \frac{E_{cm}}{2}$ since $m_{e^+} = m_{e^-}$.
  Since photons have no mass, they must each have the same energy to have equal and opposite momenta. So $E_{e^+} = E_{e^-} = E_\gamma$ and they all equal to $E_{cm}/2$
- If we think in terms of masses the final state has mass sum of zero. i.e. the initial masses are destroyed and the rest mass energy has been released. This means the reaction can proceed even if the electron and position have negligible KE.

- Reaction thresholds
  - Another case is $e^+ + e^- \rightarrow \mu^+ + \mu^-$ where muons have a much larger mass. This means the KE of initial particles are needed to form final particles. The minimum value of E_{μ^-} and E_{μ^+} is $m_\mu c^2$ so $E_{cm} \geq 2m_\mu c^2$
  - If there are N final particles created then the threshold will be $E_{cm} = \sum_i^N m_i c^2$. When $N=1$, $E_{cm} = Mc^2$ but the single particle cannot have momentum since it is conserved here it can only be created when $E_{cm} = Mc^2$. This reaction is called a "resonance"

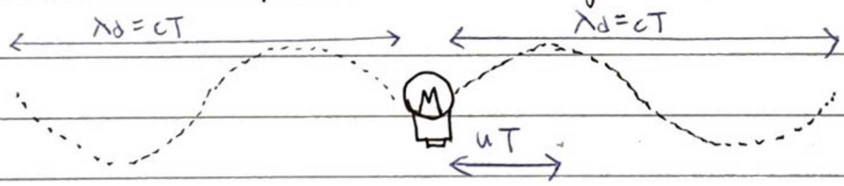
8th June 2022

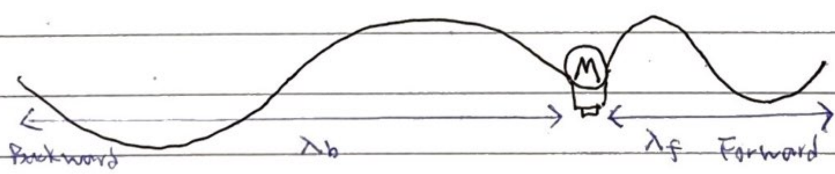
Special Relativity Lecture 10 The relativistic Doppler effect

– Time dilation

- Consider a relativistic object emitting purely perpendicular light to ~~the~~ its direction of motion. Classically there would be no Doppler shift because that is when the "ambulance" is closest to you and you hear the "true" frequency.
- However we know that the time is dilated for a moving object, which is slowed relative to the observer. If an object emits light of frequency f in its rest frame, it will actually emit light of frequency $f_d = f/\gamma_u$ when moving perpendicularly to an observer, hence the observer will always measure a lower frequency and higher wavelength i.e. "red-shifted".

– The stretch or squeeze of wavelengths





- Assume light source moving with speed u along x. Emitting one wavelength will take time equal to one period $T = \frac{1}{f_d}$ (f_d because observer's frame). During this time the leading part of the wave would have gone a distance cT while the source will have gone uT hence the wavelength will be squashed to $(c-u)T$. $\lambda_f = (c-u)T = \frac{c-u}{f_d} = \frac{c\gamma_u(1-\beta_u)}{f} = \gamma_u(1-\beta_u)\lambda$

$$f_f = \frac{c}{\lambda_f} = \frac{c}{\gamma_u(1-\beta_u)\lambda} = \frac{f}{\gamma_u(1-\beta_u)}$$

Similarly if light is going in the $-x$ direction then λ fits into $(c+u)T$ so

$$\lambda_b = \gamma_u(1+\beta_u)\lambda \quad \text{and} \quad f_b = \frac{f}{\gamma_u(1+\beta_u)}$$

$$\gamma_u(1-\beta_u) = \frac{1-\beta_u}{\sqrt{1-\beta_u^2}} = \sqrt{\frac{(1-\beta_u)^2}{(1-\beta_u)(1+\beta_u)}} = \sqrt{\frac{1-\beta_u}{1+\beta_u}}$$

$$\gamma_u(1+\beta_u) = \frac{1+\beta_u}{\sqrt{1-\beta_u^2}} = \sqrt{\frac{1+\beta_u}{1-\beta_u}}$$

$$\lambda_f = \lambda\sqrt{\frac{1-\beta_u}{1+\beta_u}} \quad \text{and} \quad f_f = f\sqrt{\frac{1+\beta_u}{1-\beta_u}}$$

$$\lambda_b = \lambda\sqrt{\frac{1+\beta_u}{1-\beta_u}} \quad \text{and} \quad f_b = f\sqrt{\frac{1-\beta_u}{1+\beta_u}}$$

with β_u positive as defined $f_f > f$ which is always a blue-shift $f_b < f$ which is always a red-shift. This shows that the wavelength squeeze always overcome the time dilation effect

- Doppler shift from four-momentum
  - Consider a photon emitted in the source rest frame in x. The photon has frequency f and energy $E = hf$ and $pc = \frac{hc}{\lambda} = hf$ as well. We can move the frame to a source moving frame using Lorentz transformation. We Lorentz transform in the negative direction to make u positive $\rightarrow$ β_u positive :

    $E' = \gamma(E - \beta pc) = \gamma_u(E + \beta_u pc) = \gamma_u(hf + \beta_u hf) = hf\gamma_u(1+\beta_u)$

    since $E' = hf_f$: $f_f = f\gamma_u(1+\beta_u) = \frac{f}{\gamma_u(1-\beta_u)}$ which is the same

    putting negative pc will be equivalent of calculating for f_b.
  - We will check when the motion is transverse to the photon and the only effect is time dilation. This means we need non-zero $p_y' = p_y$ and we need $p_x' = 0$ i.e. the Lorentz transform of $p_x = 0$

    $p_x' c = \gamma(p_x c - \beta E) = \gamma_u(p_x c + \beta_u E) = 0$ so $p_x c = -\beta_u E$

    $E' = \gamma(E - \beta p_x c) = \gamma_u(E + \beta_u p_x c) = \gamma_u(E - \beta_u^2 E) = \frac{E}{\gamma_u}$

    since $E' = hf_d$ then $f_d = \frac{f}{\gamma_u}$