A Journey Through Transmission Lines:
Understanding Group and Phase Velocity

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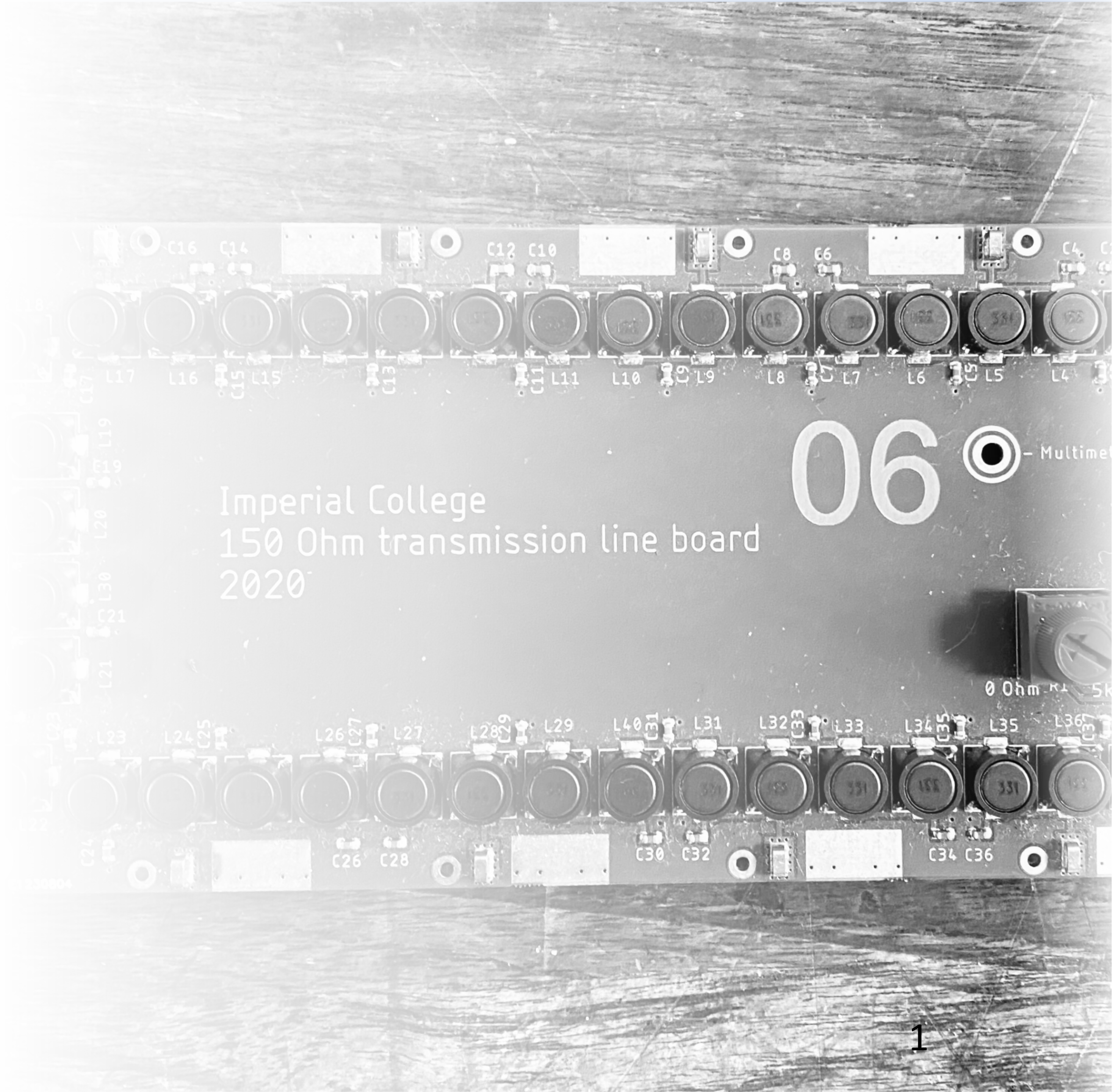
Outline:

1. Introduction
2. Methods
3. Analysis & Results
4. Conclusion



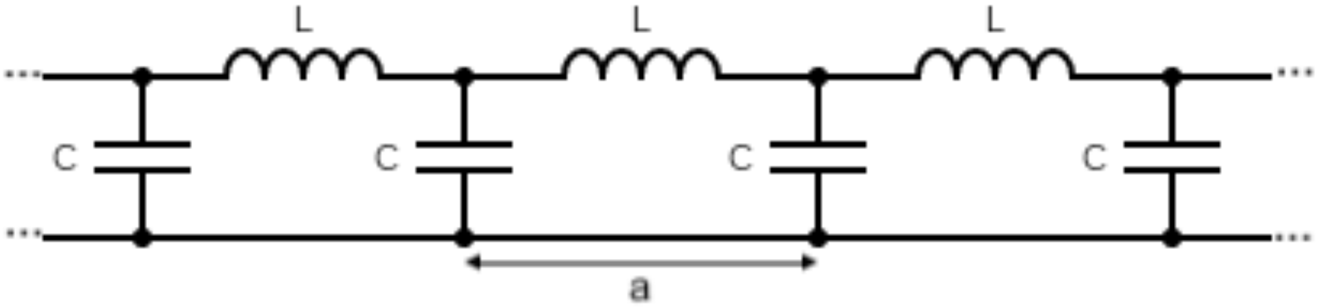
I. Introduction

- Dispersion relation: Relationship between angular frequency(ω) and angular wavenumber (k) [1]
- The dispersion relation depends only on the physics of the infinite system [2]
- Phase and Group velocity can be derived from the dispersion relation[1]

Dispersion Relation:

$$\omega^2 = \frac{4}{LC} sin^2 \frac{ka}{2}$$

*The mode of the system is proportional to $e^{\pm ikx}$



1. Figure of lumped transmission line

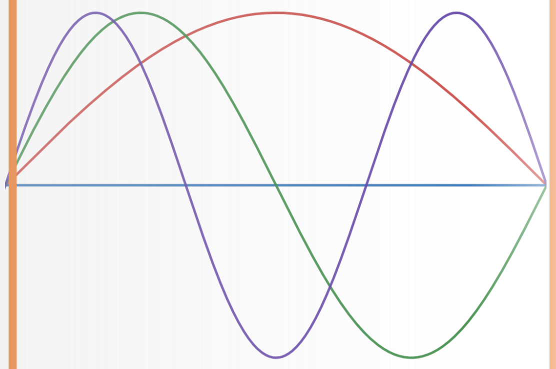
Lissajous figure method for sine waves and the Fourier analysis of rectangular waves can be used to explore frequency and phase information [3]

[1] Tamer Bécherrawy (2013). *Mechanical and Electromagnetic Vibrations and Waves*. John Wiley & Sons, Ch.4.6.
[2]Markos, P. and Soukoulis, C. (2008). Wave propagation: from electrons to photonic crystals and left-handed materials. *Choice Reviews Online*, 46(04). doi:10.5860/choice.46-2147.
[3]Cerna, M. and Harvey, A. (n.d.). *The Fundamentals of FFT-Based Signal Analysis and Measurement*. National Instruments.

II. Method

Group and Phase velocity

- The dispersion relation is obtained by treating the system as an infinite space translation invariant [4]
- Eqn. 2 shows Phase velocity expressed in frequency, length and harmonic number
- Using the Eqn. 3 and Eqn. 4, the theoretical phase velocity can be expressed by Eqn. 5 [5]
- Eqn.6 shows the group velocity, which is the k derivative of ω [6]



5. First 3 harmonics on L

$$V_p = f\lambda \quad (1)$$

$$\lambda = \frac{2L}{n}, k = \frac{2\pi}{\lambda}$$

$$V_p = \frac{2Lf}{n} \quad (2)$$

$$V_p = \frac{2Lf}{n} \cdot \frac{\pi}{\pi} = \frac{\omega}{k} \quad (3)$$

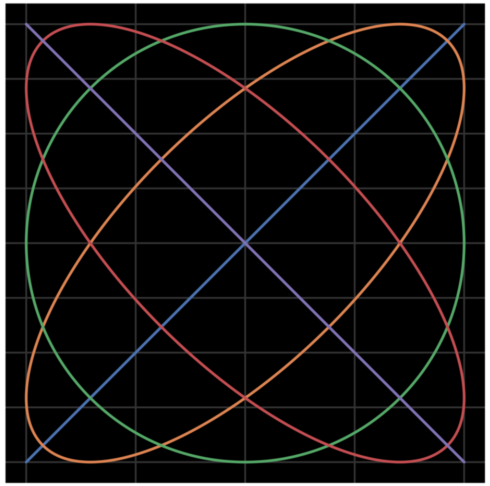
$$\omega^2 = \frac{4}{LC} sin^2 \frac{ka}{2} \quad (4)$$

$$V_p = \frac{\omega}{k} = \frac{2}{k\sqrt{LC}} sin \frac{ka}{2} \quad (5)$$

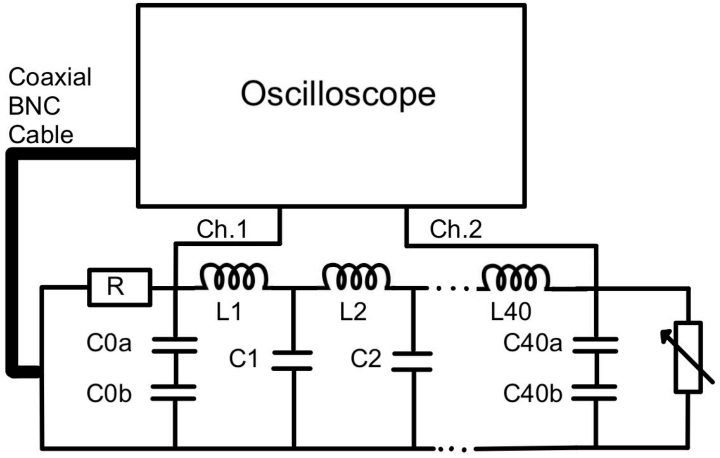
$$V_g = \frac{d\omega}{dk} = \frac{a}{\sqrt{LC}} cos \frac{ka}{2} \quad (6)$$

[4] Georgi, H. (1993). *The physics of waves*. Englewood Cliffs, N.J.: Prentice Hall, pp.107–116.
[5]Kittel, C., Mceuen, P. and Wiley, J. (2019). *Introduction to solid state physics*. Hoboken, Nj: John Wiley & Sons.
[6]Léon Brillouin (2013). *Wave Propagation and Group Velocity*. Academic Press, pp.15–25.

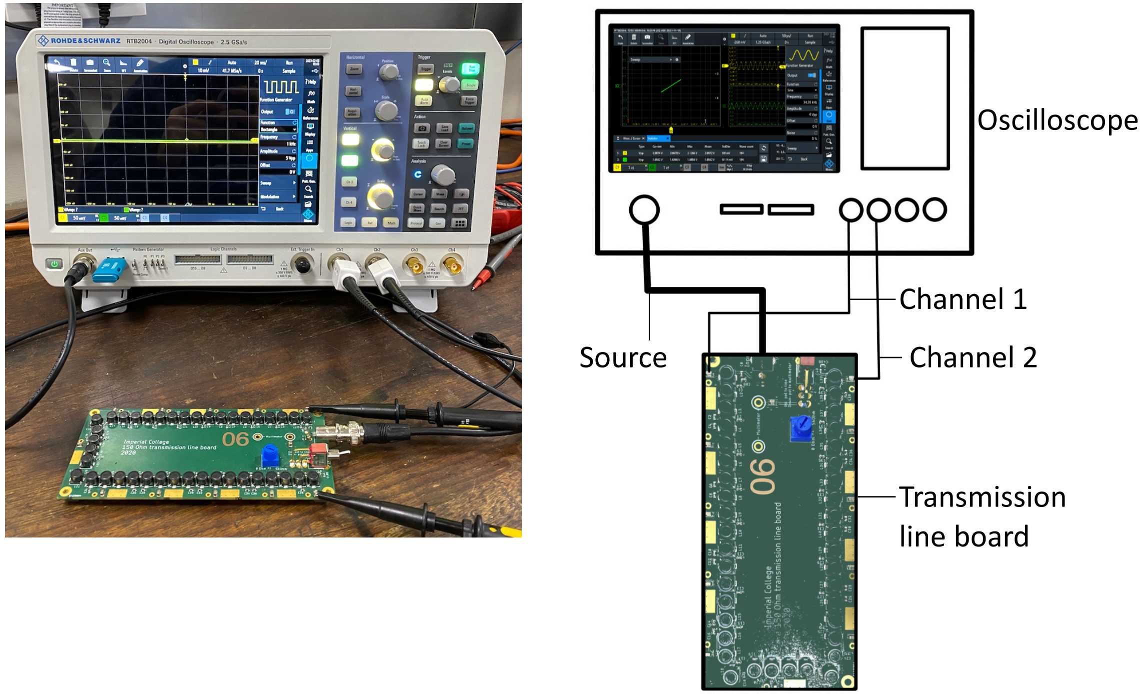
II. Method



2. Lissajous Figure



3. Schematic Diagram



4. Experimental setup

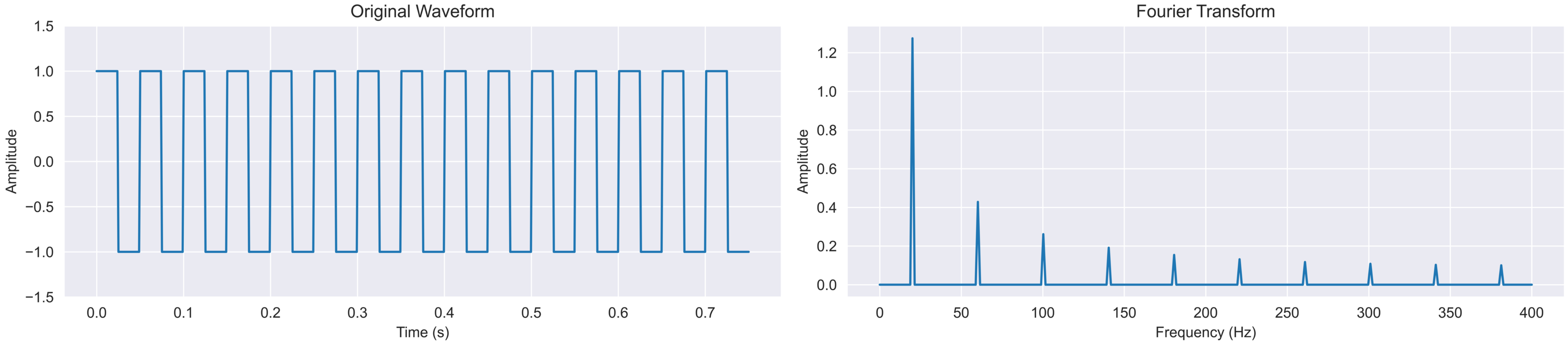
Lissajous figure method

1. Starting from 20 Hz, adjust frequency until a line is shown on figure, record frequency
2. Increase the frequency until a line of opposite gradient forms and take data
3. Repeat until the gradient of the line becomes close to 0
4. Group velocity can be calculated by $\frac{\Delta\omega}{\Delta k}$ for discontinuous data

II. Method

Fourier Analysis

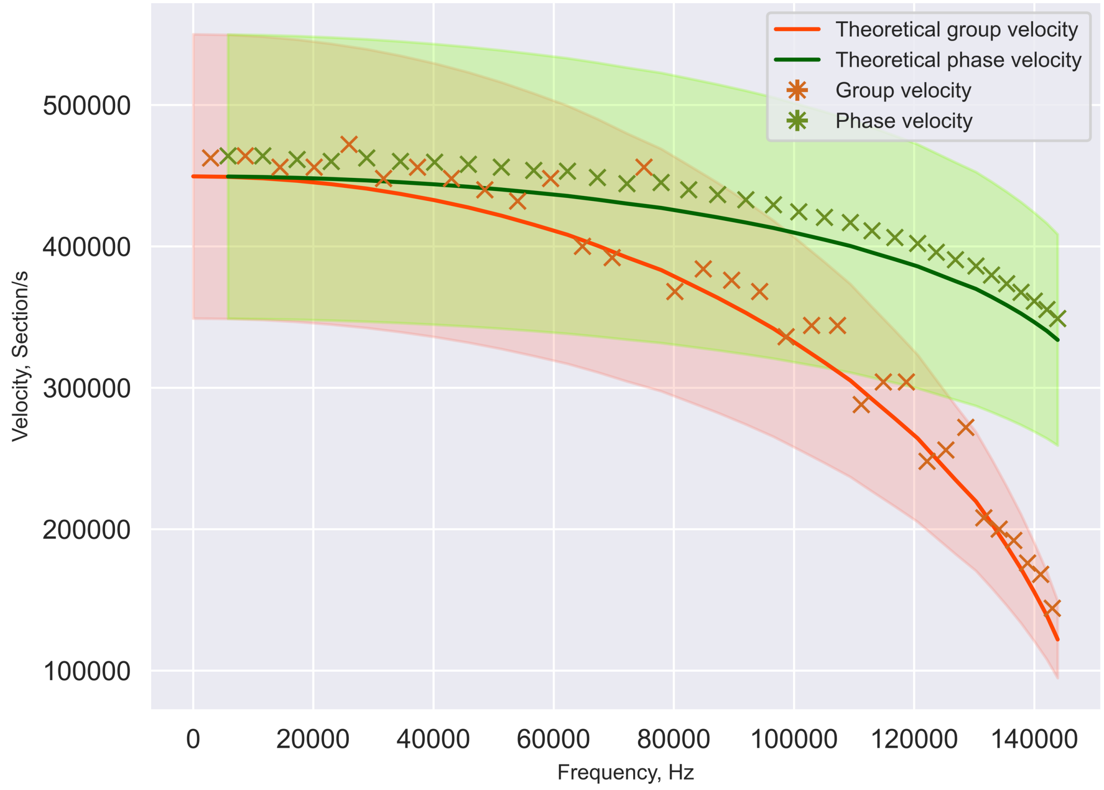
- Rectangular waves of 1, 2, and 3 kHz are generated
- The waveform at section 0 and 40 are saved as CSV files
- The time interval spans 200 ms from the origin
- Phase for each harmonic frequency are be calculated using arctan2 function



6. Example of FFT

III. Analysis & Results

- 34 data points for frequency of phase velocity
- Frequency range: 20 to 143900 Hz with a systematic error of $\pm 50\ Hz$
- The percentage uncertainties of inductance and capacitance are 20% and 10% respectively
- The measured values are within the uncertainty range
- Both phase and group velocity data follows the general shape of their theoretical value but group velocity has more variance
- Having more data points for phase velocity will increase accuracy of group velocity
  - Can be achieved by using the Fourier transform method to find information about phase and frequency, $k = \frac{\Delta\phi}{L}$ [3]



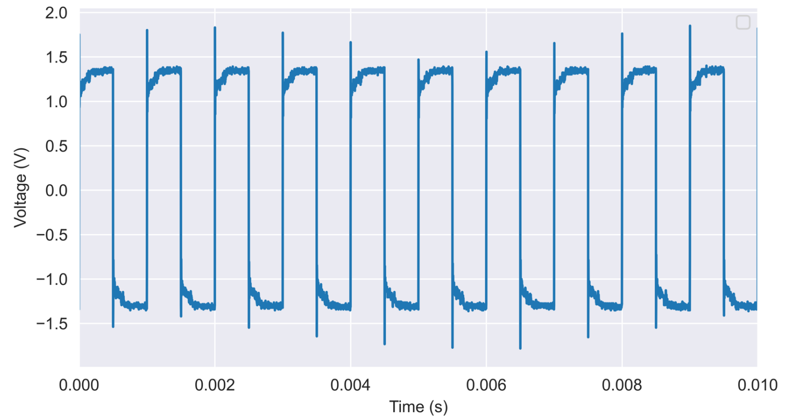
7. Plot of group and phase velocity, as well as their theoretical values

[3]Cerna, M. and Harvey, A. (n.d.). *The Fundamentals of FFT-Based Signal Analysis and Measurement*. National Instruments.

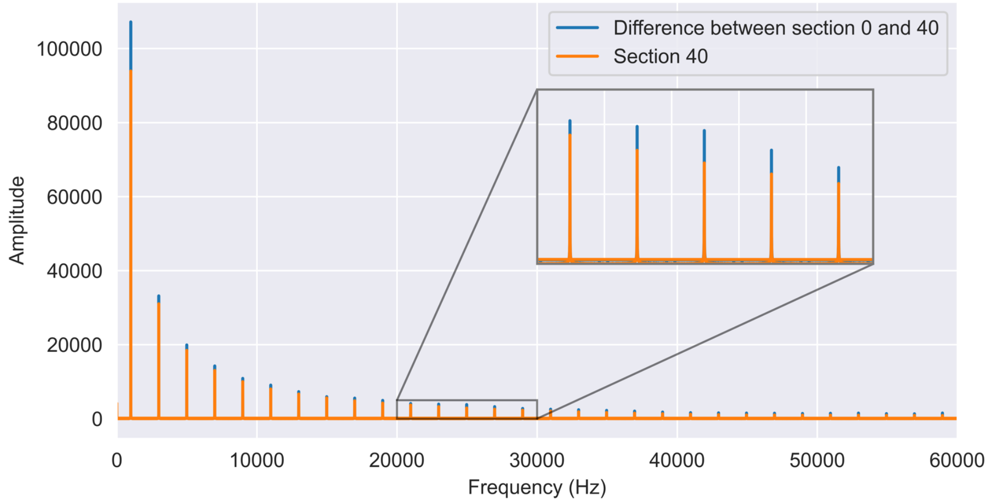
III. Analysis & Results

Fourier Analysis of a rectangular wave

- Around 130000 data points with sampling rate 540540 Hz
- A Hanning window is applied [7]
- Uncertainty depends on signal length, frequency content, and noise [7]
- The frequency range used for analysis are below the Nyquist frequency (half of sampling rate) [7]
- Peaks are identified and recorded
- Gives more accurate result than FFT on oscilloscope due to having a larger range before being truncated



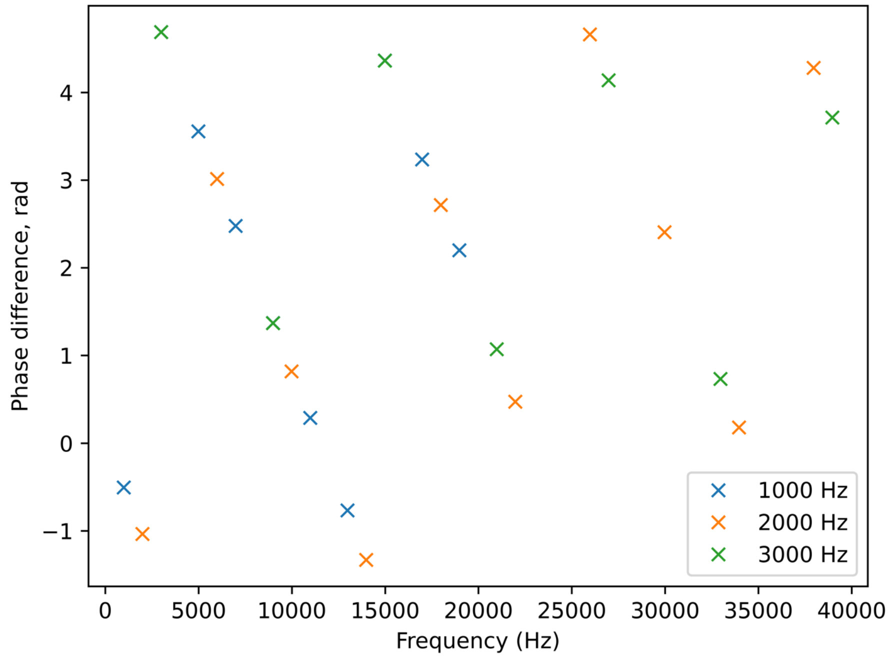
8. The plot of rectangular wave data from 0 to 0.01s



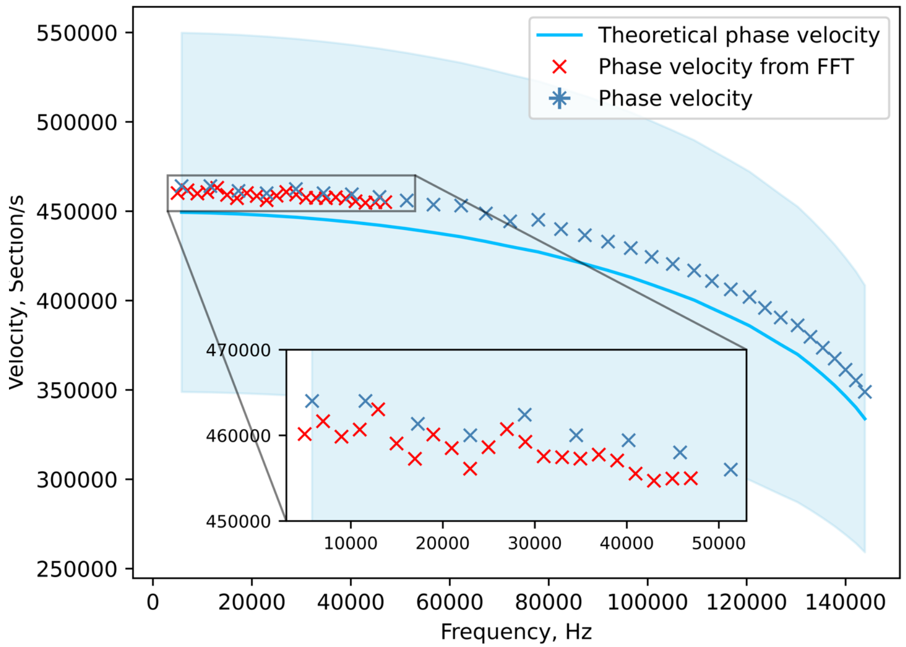
9. The plot of rectangular wave data from 0 to 0.01s

[7]E. Oran Brigham (1988). *The Fast Fourier Transform and Its Applications*. Pearson.

III. Analysis & Results



10. Plot of phase difference against different frequency



11. Phase velocity from FFT compared to phase velocity from Lissajous figures

- Phase differences between harmonics recorded
- Due to range of arctan2, phase needs to be shifted
- $V_p = \frac{\omega}{k} = \frac{2\pi f L}{\Delta\phi}$, more data are obtained
- More precise than the Lissajous figures method
- A pulse could be used to obtain more phase data[8]

[8]Rouphael, T.J. (2009). *RF and digital signal processing for software-defined radio : a multi-standard multi-mode approach*. Amsterdam ; Heidelberg: Elsevier.

IV. Conclusions

1. Both methods can give phase and group velocities in accordance to theoretical values
2. Both Phase and group velocity decreases as frequency increases; group velocity decreases at a faster rate
3. Frequency and phase information can be extracted by Fourier transforming a rectangular wave
4. Phase difference and frequency can be used to calculate phase and group velocity

Improvements & Discussion

1. Investigate the systematic error present in the circuit board
2. Use a pulse instead of rectangular waves to obtain more data
3. Estimate the uncertainty of the FFT data (algorithm, noise, simulation)