Ch.2 Fields and Forces

Coulomb - action at a distance?

Stationary charge q_1 and position $\underline{r}_1$ produces a force on a charge q_2 at $\underline{r}_2$

$$\underline{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^3} \underline{r}_{12}$$

where $\underline{r}_{12} = r_2 - r_1$ and $r_{12} = |\underline{r}_{12}|$, and $\hat{\underline{r}}_{12} = \frac{\underline{r}}{r_{12}}$

Unsatisfactory as it implies action at a distance.

Postulate the existence of a field, set up by q_1 and q_2, responds to the field locally. The electric field at a point $\underline{r}$ due to a charge that has always been at $\underline{r}_1$ is

$$\underline{E}(\underline{r}) = \frac{q_1}{4\pi\epsilon_0 |\underline{r}-\underline{r}_1|^3} |\underline{r}-\underline{r}_1|$$

If q_1 moves, according to relativity, nothing can propagate faster than light, therefore it takes a time at least $|\underline{r} - \underline{r}_1(t)|/c$ for $\underline{E}$ to respond to the changes in the position of q_1.

- How moving charges create the field (Maxwell)
- How fields propagate (Maxwell)
- How fields interact with particles (Lorentz force)

Maxwell Equations review

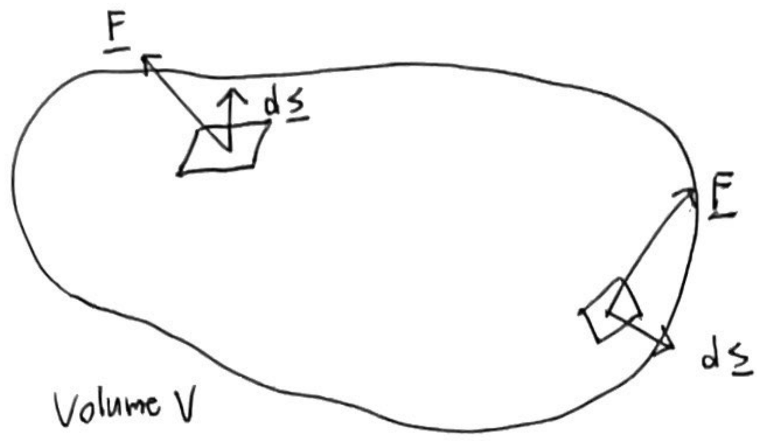
$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \; ; \; \underline{\nabla} \cdot \underline{B} = 0$$

Divergence Theorem:

consider a vector function $F(r)$. when integrated over a closed surface S enclosing a volume V, F satisfies

$$\int_V \underline{\nabla} \cdot \underline{F} \, dV = \oint_S \underline{F} \cdot d\underline{S}$$

Divergence Theorem (or Gauss's Theorem)



$$\int_V \underline{\nabla} \cdot \underline{F} \, dV = \oint_S \underline{F} \cdot d\underline{S}$$

Gauss's law is that the flux of E through a closed surface S is equal to the total charge in the volume/ϵ_0

$$\oint_S \underline{E} \cdot d\underline{S} = \frac{1}{\epsilon_0} \sum_i q_i$$

charge density $\rho(r)$

The total charge in a volume dV is $dq = \rho(r) dV$, and the total charge is $q = \int \rho(r) dV$. By Gauss's law, then using divergence theorem

$$\oint_S \underline{E} \cdot d\underline{S} = \frac{1}{\epsilon_0} \int \rho(r) dV = \int_V \underline{\nabla} \cdot \underline{E} \, dV$$

i.e. $$\int_V \left[\underline{\nabla} \cdot \underline{E} - \frac{\rho(r)}{\epsilon_0} \right] dV = 0$$

This holds for arbitrary volumes V, hence

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho(r)}{\epsilon_0}$$

The integral law for magnetic field is

$$\oint_S \underline{B} . d\underline{S} = 0$$

which implies no magnetic charges. By divergence theorem

$$\int_V \underline{\nabla} \cdot \underline{B} \, dV = 0$$

Since it holds for any V,

$$\underline{\nabla} \cdot \underline{B} = 0$$

Now consider the other two Maxwell Equations

$$\underline{\nabla} \cdot \underline{E} = -\partial_t \underline{B} \quad , \quad \underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

∂_t is $\partial/\partial t$, $\underline{J}$ is current density.

Stoke's' Theorem

When integrated along closed loop C spanned by a surface S, any well-behaved vector function satisfies

$$\int_S \underline{\nabla} \times \underline{F} \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{l}$$

Current density: The current flowing across surface element $d\underline{S}$ is $dI = \underline{J} \cdot d\underline{S}$, so $\underline{J}$ is the flux of charge per unit area ($C\,s^{-1}\,m^{-2}$)

Any surface S that terminates on C is fine.

Ampère's law is empirical

$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I$$

Since $I = \int_S \underline{J} \cdot d\underline{S}$ and using Stokes' ~~law~~ Theorem

$$\int_S \underline{\nabla} \times \underline{B} \cdot d\underline{S} = \mu_0 \int_S \underline{J} \cdot d\underline{S}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

Faraday's Law

$$\frac{d\Phi}{dt} = -\mathcal{E} \equiv -\oint_C \underline{E} \cdot d\underline{l}$$

The magnetic flux is $\Phi = \int_S \underline{B} \cdot d\underline{S}$, so

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \underline{B} \cdot d\underline{S} = \int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S}$$

The last step works because S is not changing in time.

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The last step works because S is not changing in time.

Using Stokes theorem, $\oint_C \underline{E}\cdot d\underline{l} = \int_S \underline{\nabla}\times\underline{E}\cdot d\underline{S}$, so we find that

$$\int_S (\partial_t \underline{B} + \underline{\nabla}\times\underline{E})\cdot d\underline{S} = 0$$

$$\underline{\nabla}\times\underline{E} = -\partial_t \underline{B}$$

Displacement current

Conservation of charge

Total charge in an arbitrary but fixed V enclosed by a surface S is $\int_V \rho(r)\,dV$. Charge leaving an area $d\underline{S}$ per second is $\underline{J}\cdot d\underline{S}$, so rate of loss of charge is

$$\oint_S \underline{J}\cdot d\underline{S}$$

which we equate to the rate of change of the charge enclosed

$$\frac{d}{dt}\int_V \rho(\underline{r})\,dV = -\oint_S \underline{J}\cdot d\underline{S}$$

Since V is fixed by construction, we can take the time derivative inside, where it becomes a partial derivative

$$\int_V \frac{\partial \rho(r)}{\partial t}\,dV = -\oint_S \underline{J}\cdot d\underline{S} = -\int_V \underline{\nabla}\cdot\underline{J}\,dV$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0$$

From Ampère's law we find that

$$\mu_0 \underline{\nabla}\cdot\underline{J} = \underline{\nabla}\cdot(\underline{\nabla}\times\underline{B}) = 0$$

Ampère's law must be incomplete. If we add N to Ampère's law

$$\underline{\nabla}\times\underline{B} = \mu_0\underline{J} + \underline{N}$$

$$0 = \underline{\nabla}\cdot(\underline{\nabla}\times\underline{B}) = \mu_0\underline{\nabla}\cdot\underline{J} + \underline{\nabla}\cdot\underline{N}$$

So $\underline{\nabla}\cdot\underline{J} = -\underline{\nabla}\cdot\underline{N}/\mu_0$, and conservation of charge requires

$$\partial \quad \frac{1}{\ }\underline{\nabla}\cdot\underline{N}$$

Note that the first Maxwell equation allows us to relate ρ to $\underline{E}$

$$\frac{\partial \rho}{\partial t} = \epsilon_0 \frac{\partial (\underline{\nabla} \cdot \underline{E})}{\partial t} = \epsilon_0 \underline{\nabla} \cdot \frac{\partial \underline{E}}{\partial t}$$

$$\epsilon_0 \mu_0 \underline{\nabla} \cdot \frac{\partial \underline{E}}{\partial t} = \underline{\nabla} \cdot \underline{N}$$

we can choose $\underline{N} = \epsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$

Ampére's law needs to be modified to

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \epsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$$

Ampére-Maxwell equation integral form

$$\int_S (\underline{\nabla} \times \underline{B}) \cdot d\underline{s} = \mu_0 \int_S \underline{J} \cdot d\underline{s} + \epsilon_0 \mu_0 \int_S \frac{\partial \underline{E}}{\partial t} \cdot d\underline{s}$$

$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S \underline{E} \cdot d\underline{s}$$

We need to supplement the Maxwell equations for the fields with an equation for how charges respond to local fields. The Lorentz Law:

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B}) = F$$

This equation is valid for relativistic motion.

Ch3. E and B from charge and current sheets

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0} \quad ①$$

$$\nabla \cdot \underline{B} = 0 \quad ②$$

$$\nabla \times \underline{E} + \frac{\partial B}{\partial t} = 0 \quad ③$$

$$\nabla \times \underline{B} - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 \underline{J} \quad ④$$

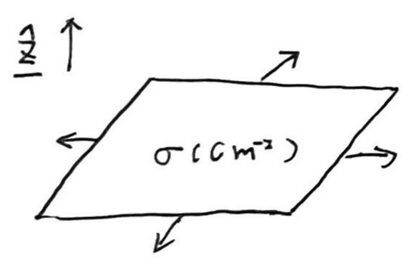
- The RHS of ① and ④ are sources of fields ($\underline{E}, \underline{B}$)
- Maxwell's equations are supplemented by Lorentz force law, which gives the response of charged particles to the (local) fields

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B}) \qquad \text{True relativistic}$$

Solving Maxwell's equations

$\underline{E}$ & $\underline{B}$ from charge and current sheets (static cases)

$\underline{E}$ from a sheet of charge with charge surface density σ (coulomb/m^2)

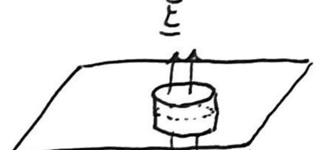


infinite

By symmetry, $\underline{E}$ is in the $\pm z$ direction

$$\underline{E} = E(z)\,\hat{\underline{z}}$$

Gauss $\oint_S \underline{E} \cdot d\underline{s} = \frac{q}{\epsilon_0}$ $\qquad$ q inclosed in V, area A



$$q = \sigma A$$

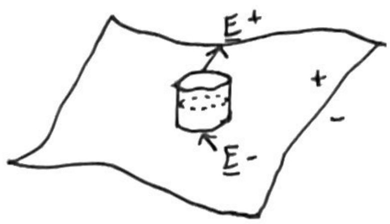
$$\oint_S \underline{E} \cdot d\underline{s} = E(z)A - E(-z)A = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow E(z) - E(-z) = \frac{\sigma}{\epsilon_0}$$

Symmetry $\rightarrow$ $E(-z) = -E(z)$

$$E(z) = \frac{\sigma}{2\epsilon_0}$$

More general : charge density on a thin sheet

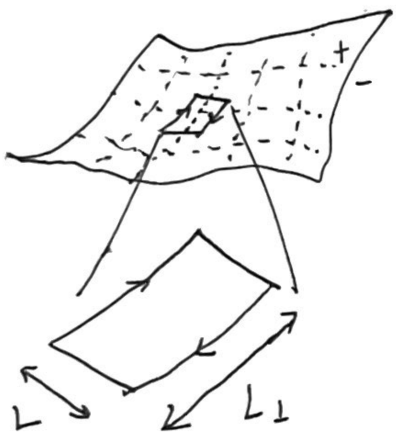


$\cancel{\oint \underline{E}\cdot d\underline{s} = E_{\parallel}^{+}}$ (parallel to normal)

$$\oint \underline{E}\cdot d\underline{s} = E_{\parallel}^{+}A - E_{\parallel}^{-}A = \frac{\sigma A}{\epsilon_0}$$

no contribution from sides if $l \to 0$ (infinitesimal)

Perpendicular $\underline{E}$



$$\oint \underline{E}\cdot d\underline{l} = \int_S \underline{\nabla}\times\underline{E}\cdot d\underline{s} = -\int_S \frac{\partial B}{\partial t}\cdot d\underline{s}$$

$\to 0$ as $L \to 0$

Provided $\frac{\partial B}{\partial t}$ is finite

$L \to 0 \Rightarrow$ no contribution of LHS from "Legs"

$$\Rightarrow E_{\perp}^{+}L_{\perp} - E_{\perp}^{-}L_{\perp} = 0$$

$\underline{E} \nearrow \Rightarrow \underline{E}_{\perp}$ is continuous

Magnetic Field

$\underline{\nabla}\cdot\underline{B} = 0 \Rightarrow \underline{B}_{\parallel}$ is continuous

$B_{\perp}$

I (current per unit length)

$$\oint_c \underline{B}\cdot d\underline{l} = \int_S (\underline{\nabla}\times\underline{B})\cdot d\underline{s} = \int_S \mu_0 \underline{J}\cdot d\underline{s}$$

(static case)

Loop at right angles to current

$$\int \mu_0 \underline{J}\cdot d\underline{s} = \mu_0 \times \text{charge / see crossing loop}$$

$$= \mu_0 I L_{\perp}$$

$$B_{\perp\perp}^{+}L_{\perp} - B_{\perp\perp}^{-}L_{\perp} = \mu_0 I L_{\perp}$$

$\Delta \underline{B}_{\perp\perp} = \mu_0 I$

$\Delta B_{\perp\parallel} = 0$

Perpendicular to normal

Parallel to current

From Maxwell's Equations to Wave Equations

4.1 Wave Equations in vacuum

If we remove $\underline{B}$ from Maxwell's equations, in the absense of any sources. From Faraday's law,

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\underline{\nabla} \times \partial_t \underline{\nabla} \times \underline{B} = -\partial_t \epsilon_0 \mu_0 \partial_t \underline{E}$$

For any vector field $\underline{F}(\underline{r})$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{F}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{F}) - \underline{\nabla}^2 \underline{F}$$

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F}) = 0$$

For any scalar field $\psi(\underline{r})$

$$\underline{\nabla} \times (\underline{\nabla} \psi) = 0$$

$$\underline{\nabla} \cdot (\psi \underline{F}) = \psi \underline{\nabla} \cdot \underline{F} + \underline{F} \cdot \underline{\nabla} \psi$$

Now $\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{E}) - \underline{\nabla}^2 \underline{E}$, and without sources $\underline{\nabla} \cdot \underline{E} = 0$

$$\underline{\nabla}^2 \underline{E} - \epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2} = 0$$

This is a vector wave equation for $\underline{E}$. In cartesian coordinates, this is just three scalar wave equations for each component, E_x, E_y, E_z. For the cartesian case, the equation for E_x (E_y, E_z) is

$$\underline{\nabla}^2 E_x - \epsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

Let's assume for now that $\underline{E}$ is in the x direction, and that quantities depend on z and t only

$$\frac{\partial^2 E_x}{\partial z^2} - \epsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

with the general solution

$$E_x = f(z - ct) + g(z + ct)$$

where f and g are arbitrary functions, and

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

We can also eliminate $\underline{E}$ from Maxwell's equations. ~~To this~~

$$\underline{\nabla}^2 \underline{B} - \epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2} = 0$$

4.1.2 Components of E and B parallel to propagation direction

With the only spatial variation being in the z direction,

$$\underline{\nabla} \cdot \underline{E} = \frac{\partial E_z}{\partial z} = 0$$

since there are no charges. Hence $E_z(z,t)$ can depend on t only, not on z. Assuming $E_z \to 0$ as $z \to \pm\infty$, we obtain $E_z(z,t) = 0$ there is no electric field in the propagation direction.

For $\underline{B}$ we also have

$$\underline{\nabla} \cdot \underline{B} = \frac{\partial B_z}{\partial z} = 0$$

By similar argument $B_z(z,t) = 0$

Therefore we see that $\underline{E}$ and $\underline{B}$ are perpendicular to the direction of propagation, z.

Let's determine the direction of $\underline{B}$, using Ampère:

$$\underline{\nabla} \times \underline{B} = \frac{1}{c^2} \partial_t \underline{E}$$

Hence

$$\begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 0 & 0 & \partial_z \\ B_x & B_y & B_z \end{vmatrix} = \frac{1}{c^2} \partial_t \underline{E}_x \hat{i}$$

Hence $-\frac{\partial B_y}{\partial z} = \frac{\partial E_x}{\partial t} \frac{1}{c^2}$ and so B_y is non-zero. $E_x \neq 0$, $B_y \neq 0$ corresponds to a wave linearly polarized in the x direction.

$\underline{E}$, $\underline{B}$ and the vector wave direction are mutually orthogonal.

Simple solutions

Sinusoidal solutions to the wave equation exists, e.g.

$$E_x(z,t) = E_0 \cos(kz - \omega t),$$

where k is the wavenumber, ω the angular frequency, E_0 the amplitude, and $\phi = kz - \omega t$ is the phase. Wave crests move at a speed given by $\phi = kz - \omega t = 0$, hence $z/t = \omega/t$ so the phase speed is

$$V_{phase} = \frac{\omega}{k}$$

The wavelength is $\lambda = 2\pi/k$

4.2 Complex representation

Sinusoidal solution polarized in x direction with propagation in the z direction is

$$E_x = E_0 \cos(kz - \omega t + \psi)$$

where ψ is an arbitrary (real) phase offset and E_0 is real. This is a travelling wave in the $+z$ direction.

Let us write this in complex form, noting $\cos x = \mathrm{Re}(e^{ix})$

$$E_0 \cos(kz - \omega t + \psi) = \mathrm{Re}\{E_0 \exp(i(kz - \omega t + \psi))\} \equiv \mathrm{Re}(\underline{\tilde{E}}).$$

$$= \mathrm{Re}\{\tilde{E} \exp[i(kz - \omega t)]\}.$$

where we define the complex amplitude:

$$\tilde{E}_0 = E_0 \exp(i\psi)$$

We similarly consider a complex $\underline{\tilde{B}}$, noting that the Re and Im parts both satisfy the Maxwell equations, which are linear, so the sum also obeys the equations. For example, with $\underline{\tilde{B}} = \underline{\tilde{B}}_r + i\underline{\tilde{B}}_i$ and similarly for $\underline{\tilde{E}}$, the separate components satisfy

$$\underline{\nabla} \times \underline{\tilde{B}}_r = \epsilon_0 \mu_0 \partial_t \underline{\tilde{E}}_r$$

and $$\underline{\nabla} \times \underline{\tilde{B}}_i = \epsilon_0 \mu_0 \partial_t \underline{\tilde{E}}_i$$

$$\rightarrow \underline{\nabla} \times (\underline{\tilde{B}}_r + i\underline{\tilde{B}}_i) = \epsilon_0 \mu_0 \partial_t (\underline{\tilde{E}}_r + i\underline{\tilde{E}}_i)$$

$$\rightarrow \underline{\nabla} \times \underline{\tilde{B}} = \epsilon_0 \mu_0 \partial_t \underline{\tilde{E}}$$

Relating complex fields $\underline{\tilde{E}}$ and $\underline{\tilde{B}}$

Electromagnetic waves in vacuum are transverse

We can give a more elegant argument to show that EM waves are transverse, and that $\underline{E}$, $\underline{B}$, and $\underline{k}$ are mutually orthogonal.

For a monochromatic wave with wavevector $\underline{k}$,

$$\underline{\tilde{E}} = \underline{\tilde{E}}_0 \exp[i(\underline{k}\cdot\underline{r} - \omega t)] \qquad \underline{\tilde{B}} = \underline{\tilde{B}}_0 \exp[i(\underline{k}\cdot\underline{r} - \omega t)]$$

Note that the product rule for differentiation translates operators into multiplications:

$$\frac{\partial}{\partial t} \rightarrow -i\omega \; ; \; \frac{\partial}{\partial x_i} \rightarrow ik_i \quad \text{or} \quad \underline{\nabla} \rightarrow i\underline{k}$$

For example:

$$\partial_t \underline{\tilde{E}} = -i\omega \underline{\tilde{E}}$$

$$\underline{\nabla} \cdot \underline{\tilde{E}} = i\underline{k} \cdot \underline{\tilde{E}}$$

$$\underline{\nabla} \times \underline{\tilde{E}} = i\underline{k} \times \underline{\tilde{E}}$$

To prove the last one, consider

$$(\underline{\nabla} \times \hat{\underline{E}})_x = \partial_y \hat{E}_z - \partial_z \hat{E}_y = ik_y \hat{E}_z - ik_z \hat{E}_y = i(\underline{k} \times \hat{\underline{E}})_x$$

which generalises to the last equation above

Hence

$$\underline{\nabla} \times \hat{\underline{E}} = -\partial_t \tilde{\underline{B}} \qquad (4.26)$$

$$\rightarrow i\underline{k} \times \hat{\underline{E}} = i\omega \tilde{\underline{B}}$$

and taking the scalar product with $\hat{E}$ gives zero, so

$$\hat{\underline{E}} \cdot (\underline{k} \times \hat{\underline{E}}) = \omega \hat{\underline{E}} \cdot \tilde{\underline{B}} = 0$$

so $\hat{E}$ and $\tilde{B}$ are orthogonal. Taking the scalar product of 4.26 with $\underline{k}$ shows that $\underline{k}$ and $\tilde{\underline{B}}$ are also orthogonal, so we just need to show that $\hat{\underline{E}}$ and $\underline{k}$ are orthogonal and we are done.

In vacuum,

$$\underline{\nabla} \times \tilde{\underline{B}} = \frac{1}{c^2} \partial_t \tilde{\underline{E}}$$

$$\rightarrow i\underline{k} \times \tilde{\underline{B}} = -\frac{i\omega}{c^2} \tilde{\underline{E}}$$

and taking the scalar product with k gives

$$\underline{k} \cdot \hat{\underline{E}} = 0$$

We can also show that $c = \frac{\omega}{k}$, by taking the curl of Ampère's law

$$\underline{\nabla} \times (i\underline{k} \times \tilde{\underline{B}}) = \underline{\nabla} \times \left(-\frac{i\omega}{c^2} \hat{\underline{E}}\right)$$

$$\rightarrow i\underline{k} \times (i\underline{k} \times \tilde{\underline{B}}) = -\frac{i\omega}{c^2} (\underline{\nabla} \times \tilde{\underline{E}})$$

$$\rightarrow -\left[(\underline{k} \cdot \tilde{\underline{B}})\underline{k} - (\underline{k} \cdot \underline{k})\tilde{\underline{B}}\right] = -\frac{i\omega}{c^2} (i\omega \tilde{\underline{B}})$$

$$k^2 \tilde{\underline{B}} = \frac{\omega^2}{c^2} \tilde{\underline{B}}$$

where the expansion of the vector triple product $\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B}) \cdot \underline{C}$ is used. Also the fact that $\underline{k}$ and $\tilde{\underline{B}}$ are orthogonal. Hence

$$c = \frac{\omega}{k}$$

Let's see how $|\hat{\underline{E}}_0|$ and $|\tilde{\underline{B}}_0|$ are related. Ampère's law (dividing by i) gives

$$\underline{k} \times \tilde{\underline{B}} = -\frac{\omega}{c^2} \tilde{\underline{E}}$$

and since $\underline{k}$ and $\tilde{\underline{B}}$ are orthogonal, the magnitude of the LHS is $k|\tilde{\underline{B}}_0|$

so,

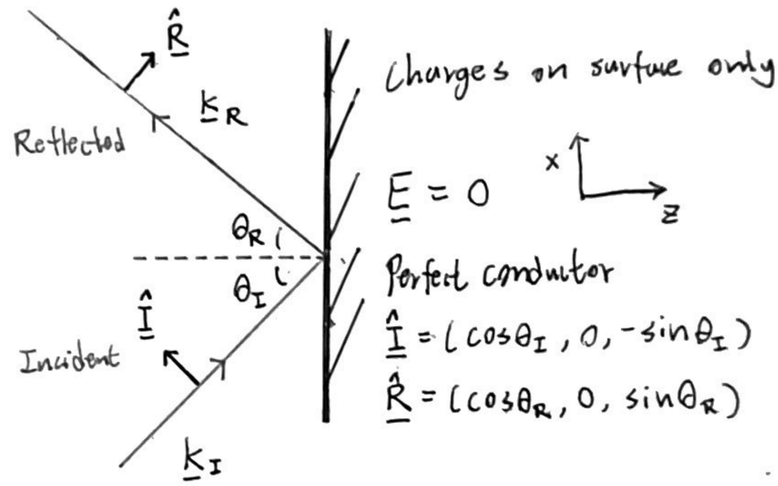
$$|\tilde{\underline{B}}_0| = \frac{\omega}{c^2 \underline{k}} |\tilde{\underline{E}}_0| = \frac{|\tilde{\underline{E}}_0|}{c}$$

So the amplitude of the $\underline{B}$ field is $\frac{1}{c}$ times that of the $\underline{E}$ field

Reflection at conductors

Let's consider what happens if a monochromatic wave of wavenumber k_I in a vacuum hits a conductor with a flat surface, at an angle θ_I to the normal, and is reflected with wavenumber k_R, at an angle θ_R to the normal.

A perfect conductor has no electric field inside. Charges may accumulate on the surface. So we have a situation with a current sheet on the boundary, and can use the boundary conditions we found before.



Maxwell's equations relate $\hat{\underline{E}}$ and $\tilde{\underline{B}}$ for incident and reflected waves:
First, the incident and reflected waves have phase dependence $\exp[i(\underline{k}\cdot\underline{r} - \omega t)]$.
If the interface is at $z=0$, then, for example, at $x=0$ and $y=0$, then to satisfy the boundary conditions for all time, we require

$$\omega_I = \omega_R$$

Since the incident and reflected waves are in vacuum, and $\omega/k = c$

$$|k_I| = |k_R| \equiv k$$

Let's consider a wave that is linearly polarized, and without loss of generality we choose k_I to be in the xz plane, and for simplicity let us choose a particular polarisation, also in the xz plane.

The incident E field is

$$\tilde{\underline{E}}_I = \hat{E}_I \hat{\underline{I}} \exp[i(\underline{k}_I \cdot \underline{r} - \omega t)]$$

where $\underline{k}_I = k_I(\sin\theta_I, 0, \cos\theta_I)$ and $\hat{\underline{I}}$ is $(\cos\theta_I, 0, -\sin\theta_I)$, so it is transverse $(\hat{\underline{I}} \times \underline{k}_I = 0)$

Let the reflected wave have some propagation direction $\underline{k}_R$, at an angle θ_R to the normal, with the electric field lying along $\hat{\underline{R}}$. Then

$$\tilde{\underline{E}}_R = \tilde{E}_R \hat{\underline{R}} \exp[i(\underline{k}_R \cdot \underline{r} - \omega t)]$$

where $\hat{\underline{R}} = (\cos\theta_R, 0, \sin\theta_R)$

Since the perpendicular component of $\bar{E}$ is continuous at the boundary and is zero inside the conductor, it is zero outside as well.

This requires

$$\exp(-i\omega t)[\tilde{E}_I \hat{\underline{I}} \exp(i\underline{k}_I \cdot \underline{r}) + \tilde{E}_R \hat{\underline{R}} \exp(i\underline{k}_R \cdot \underline{r})]_\perp = 0 \quad 4.38$$

It is convenient to look at this at $r = (X, 0, 0)$, where

$$\underline{k}_I \cdot \underline{r} = k X \sin\theta, \quad \underline{k}_R \cdot \underline{r} = k X \sin\theta_R$$

Since equation 4.38 holds for all X, the complex exponentials must agree and

$$\sin\theta_I = \sin\theta_R$$

$$\theta_I = \theta_R$$

and the complex amplitudes are the same, but of different sign:

$$\tilde{\underline{E}}_R = -\tilde{\underline{E}}_I$$

since $\hat{\underline{I}}_\perp = \cos\theta_I$ and $\hat{\underline{R}}_\perp = \cos\theta_R$

So what we find is that the wave gets reflected at the same angle

Ch.5 Energy densities and energy flux

5.1 Energy in electromagnetic field

The electric and magnetic fields get propagated to the charge q, which responds according to the Lorentz law:

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$$

where $\underline{v}$ is charge velocity

For example, a stationary charge would be accelerated by $\underline{E}$, and gain energy. Where does this energy come from?

5.1.1 Workdone on charge q

The work done is $\underline{F}(\underline{r}) \cdot d\underline{r}$ for movement through dr, so the power P delivered to the charge is:

$$P = \underline{F} \cdot \frac{d\underline{r}}{dt} = \underline{F} \cdot \underline{v}$$

From the Lorentz force,

$$P = \underline{F} \cdot \underline{v} = q[\underline{v} \cdot \underline{E} + \underline{v} \cdot (\underline{v} \times \underline{B})] = q\,\underline{v} \cdot \underline{E}$$

and the second term vanishes from triple product equation, meaning that magnetic fields do no work.

From Newton's second law, $\underline{F} = \frac{d(m\underline{v})}{dt}$

$$\underline{F} \cdot \underline{v} = m\frac{d\underline{v}}{dt} \cdot \underline{v} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\,\underline{v} \cdot \underline{E}$$

So the power delivered to the charge is equal to the rate of change of kinetic energy.

5.1.2 Ensemble of particles

The charge density of a charge q_i at position $r_i(t)$ is a Dirac Delta function

$$\rho(r,t) = q_i\,\delta(r - r_i(t))$$

and the current density is

$$\underline{J}(r,t) = q_i\,\underline{v}_i\,\delta(\underline{r} - \underline{r}_i(t))$$

where $\underline{v}_i = d\underline{r}_i/dt$ is its velocity

For a volume V containing N charges

$$\rho(\underline{r},t) = \sum_{i=1}^{N} q_i \delta(\underline{r}-\underline{r}_i(t))$$

$$\underline{J}(\underline{r},t) = \sum_{i=1}^{N} q_i \underline{v}_i \delta(\underline{r}-\underline{r}_i(t))$$

The rate of change of the total kinetic energy is

$$\sum_{i=1}^{N} \frac{d}{dt}\left(\frac{1}{2}m_i v_i^2\right) = \sum_{i=1}^{N} \int_V q_i \underline{v}_i \cdot \underline{E}(\underline{r}_i,t)\,\delta(\underline{r}-\underline{r}_i(t))\,dV = \int_V \underline{J}\cdot\underline{E}\,dV$$

So the rate at which work is done on charges by the field, per unit volume, is $\underline{J}\cdot\underline{E}$

Calculation of $\underline{J}\cdot\underline{E}$

We can use Maxwell's equations to calculate $\underline{J}\cdot\underline{E}$. Using Ampère-Maxwell,

$$\underline{\nabla}\times\underline{B} = \mu_0\underline{J} + \epsilon_0\mu_0\partial_t\underline{E}$$

and taking the scalar product with $\underline{E}$,

$$\underline{J}\cdot\underline{E} = \frac{1}{\mu_0}\underline{E}\cdot(\underline{\nabla}\times\underline{B}) - \epsilon_0(\partial_t\underline{E})\cdot\underline{E}$$

$$= \frac{1}{\mu_0}\underline{E}\cdot(\underline{\nabla}\times\underline{B}) - \partial_t\left(\frac{1}{2}\epsilon_0 E^2\right) \qquad (5\cdot10)$$

where $E^2 = |\underline{E}|^2$

5.2 Poynting's Theorem and the flux of EM energy

We can write $\underline{J}\cdot\underline{E}$ in a more obvious form, by using Faraday and some more vector calculus:

$$\underline{\nabla}\times\underline{E} = -\partial_t\underline{B}$$

and taking the scalar product with $\underline{B}/\mu_0$

$$\frac{1}{\mu_0}\underline{B}\cdot(\underline{\nabla}\times\underline{E}) = -\partial_t\left(\frac{1}{2\mu_0}B^2\right) \qquad (5\cdot12)$$

where $B^2 = |\underline{B}|^2$

If we combine these, adding the RHS and subtracting the LHS of 5.12 to 5.10,

$$\underline{J}\cdot\underline{E} = \frac{1}{\mu_0}\left[\underline{E}\cdot(\underline{\nabla}\times\underline{B}) - \underline{B}\cdot(\underline{\nabla}\times\underline{E})\right] - \frac{\partial}{\partial t}\left[\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right]$$

We can use Divergence of a vector product:

For vector fields $\underline{F}(\underline{r})$ and $\underline{G}(\underline{r})$, $\quad \underline{\nabla}\cdot(\underline{F}\times\underline{G}) \equiv \underline{G}\cdot(\underline{\nabla}\times\underline{F}) - \underline{F}\cdot(\underline{\nabla}\times\underline{G})$

So we find that

$$\underline{J} \cdot \underline{E} = -\frac{1}{\mu_0} \underline{\nabla} \cdot (\underline{E} \times \underline{B}) - \frac{\partial}{\partial t}\left[\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2\right]$$

This is Poynting's Theorem

- $W \equiv \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$ is the electromagnetic energy density per unit volume
- $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$ is the Poynting vector
- $\underline{J} \cdot \underline{E}$ is the power delivered to the charges per unit volume

The use of $\underline{S}$ to denote the Poynting vector is inconvenient, but it should be clear when it means this and when it refers to a surface.

Poynting's theorem can then be written

$$\frac{\partial W}{\partial t} + \underline{\nabla} \cdot \underline{S} = -\underline{J} \cdot \underline{E}$$

This is Poynting's theorem in differential form.

This has some similarities with the continuity equation, except that the energy in the fields is not conserved, so there is a source (or sink if $\underline{J} \cdot \underline{E} > 0$) of energy if $\underline{J} \cdot \underline{E} \neq 0$. We interpret the Poynting vector $\underline{S}$ as the flux of energy carried by the electromagnetic field.

Let's write it in integral form by integrating over an arbitrary but fixed volume V, $\underline{S} \cdot d\underline{S}$ is the flux of electromagnetic energy through a surface element $d\underline{S}$. Using the divergence theorem

$$\int_V \underline{\nabla} \cdot \underline{S} \, dV = \oint_S \underline{S} \cdot d\underline{S}$$

Hence Poynting's theorem in integral form is

$$\frac{d}{dt} \int W \, dV = -\oint_S \underline{S} \cdot d\underline{S} - \int_V \underline{J} \cdot \underline{E} \, dV$$

i.e. Rate of change of EM energy in V equals to rate of ~~change~~ flow of energy into the bounding surface minus the energy per second transferred from the EM field to charges. $\underline{S}$ is therefore interpreted as the energy flux density carried in the electromagnetic field. For a wave with wavevector $\underline{k}$, it is in the direction of $\underline{k}$.

5.2.2 Quadratic quantities and complex fields

For complex $\underline{E}$ and $\underline{B}$ fields we simply take the real part in order to obtain what we would actually measure. For quadratic quantities we need to do something different. To illustrate, consider the power per unit volume delivered from the electromagnetic field to the charges

$$P = \underline{J} \cdot \underline{E}$$

If we write $\underline{J}$ and $\underline{E}$ in terms of complex quantities ($\underline{J}$ is the real part of $\tilde{\underline{J}}$ etc.)

$$P = \frac{1}{2}(\tilde{\underline{J}} + \tilde{\underline{J}}^*) \cdot \frac{1}{2}(\tilde{\underline{E}} + \tilde{\underline{E}}^*) = \frac{1}{4}(\tilde{\underline{J}}\cdot\tilde{\underline{E}} + \tilde{\underline{J}}^*\cdot\tilde{\underline{E}}^* + \tilde{\underline{J}}\cdot\tilde{\underline{E}}^* + \tilde{\underline{J}}^*\tilde{\underline{E}})$$

For sinusoidal waves (or for a single Fourier component) the complex terms have a time-dependence $e^{-i\omega t}$, so the first term is proportional to $e^{-2i\omega t}$, which averages to zero over time. Similarly for the $\tilde{\underline{J}}^*\cdot\tilde{\underline{E}}^*$ term. The other two terms (both terms are the same) have the time-dependence cancelling out, so the time-averaged power per unit volume is

$$\langle P\rangle = \frac{1}{2}\mathrm{Re}(\tilde{\underline{J}}^*\cdot\tilde{\underline{E}})$$

5.3 Poynting examples

Plane monochromatic wave

For a wave travelling in the k direction, ~~as~~ sho we see that the Poynting vector $\underline{S} = \underline{E}\times\underline{B}/\mu_0$ is in the direction of $\underline{k}$, as we expect.

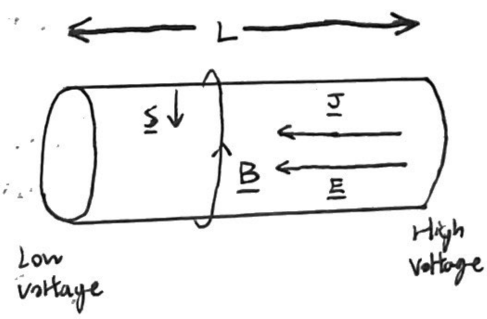
Heating a wire

We know that the power dissipated in a wire is $P = IV$, where I is current and V is the voltage drop across the wire, whose length is L. We can obtain this result by looking at the field energy crossing into the wire. Ampére's law tells us at the edge of the wire ($r=a$) the magnetic field is:

$$B = \frac{\mu_0 I}{2\pi a}$$

and it is tangential. The electric field is $\frac{V}{L}$, and lies along the wire. The Poynting vector therefore points inward at the surface, and the energy per unit area crossing the surface is

$$\underline{S} = \frac{1}{\mu_0}\underline{E}\,\underline{B} = \frac{V}{\mu_0 L}\frac{\mu_0 I}{2\pi a} = \frac{IV}{2\pi a L}$$



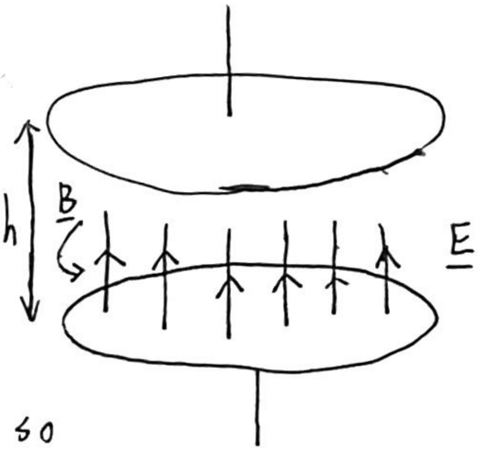
The poynting vector tells us that there is a flux of EM energy into the wire. Since the situation is in a steady state, that energy has to leave the EM field at the same rate. It does this by passing energy to the electrons at a rate $\underline{J}\cdot\underline{E}$ per unit volume, which integrates to IV within the whole volume.

The energy flow in a changing capacitor

Imagine a capacitor is slowly acquiring charge. An electric field builds up between the plates. How does the energy get there? If the separation between the plates is small, $\underline{E}$ is uniform, and builds up at a rate $\dot{\underline{E}}$. The energy in the electric field is $\epsilon_0 E^2/2$, per unit volume, so the total energy (in the volume $\pi a^2 h$) changes at a rate

$$\frac{dU_E}{dt} = \epsilon_0 E \dot{E} \times \pi a^2 h$$

How does this energy get in? Certainly not from the direction of the wires since Poynting vector is at right angles to $\underline{E}$. There is a magnetic field associated with the changing electric field, described by Ampère's law.



There are no currents between the plates so

$$\underline{\nabla} \times \underline{B} = \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

In integral form, we take a loop that sits between the plates with radius a:

$$\oint_C \underline{B} \cdot d\underline{l} = \epsilon_0 \mu_0 \int \dot{\underline{E}} \cdot d\underline{S} = \epsilon_0 \mu_0 \dot{\underline{E}}\, \pi a^2$$

$\underline{B}$ is tangential, and the LHS is $B 2\pi a$, so we find

$$B = \frac{\epsilon_0 \mu_0 a \dot{E}}{2}$$

The Poynting vector thus points inwards, through the outside cylindrical surface, with magnitude

$$S = \frac{1}{\mu_0} EB = \frac{1}{\mu_0} E \frac{\epsilon_0 \mu_0 a \dot{E}}{2} = \frac{\epsilon_0 a E \dot{E}}{2}$$

Since S is the power per unit area, the total power coming in is $2\pi a h S$ i.e.

$$\epsilon_0 E \dot{E} \times \pi a^2 h$$

Ch.6 Scalar and Vector Potentials

6.1 Electrostatics

By statics we mean solutions to time-independent problems, $\partial_t = 0$.

Maxwell's equations simplify to

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \qquad \text{electrostatics}$$

$$\underline{\nabla} \times \underline{E} = 0$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \text{magnetostatics}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

In static case $\underline{E}$ and $\underline{B}$ are completely decoupled.
The condition means that the currents are not time-varying

Maths: Curl-free fields

If $\underline{F}(\underline{r})$ is a vector field, and

$$\underline{\nabla} \times \underline{F} = 0$$

then we can always find a scalar field $\psi(r)$ such that

$$\underline{F} = \underline{\nabla}\psi$$

$\underline{F}$ also satisfies

$$\oint_C \underline{F} \cdot d\underline{l} = 0$$

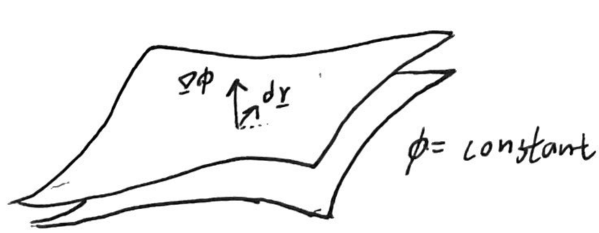
around any closed loop C. It is also true if $\underline{F} = \underline{\nabla}\psi$, then $\underline{\nabla} \times \underline{F} = 0$

6.1.2 Electrostatic Potential

Let's consider electrostatics. Because $\underline{\nabla} \times \underline{E} = 0$, we can find a scalar field $\phi(r)$ such that

$$\underline{E} = -\underline{\nabla}\phi$$

The minus is a convention. $\phi(r)$ is called the electrostatic potential or scalar potential. Note it's not unique – you can add a constant.
Constant ϕ surfaces are perpendicular to the direction of $-\underline{\nabla}\phi$.



From the chain rule, $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\underline{r} \cdot \underline{\nabla}\phi$

Hence is zero if $d\underline{r} \perp \underline{E}$, and so is the locus $\phi =$ constant

We can find ϕ by integrating $\underline{E}$:

$$\int_a^b \underline{E}\cdot d\underline{r} = -\int_a^b d\underline{r}\cdot\underline{\nabla}\phi = -\int_{\phi(a)}^{\phi(b)} d\phi = \phi(a) - \phi(b)$$

This allows potential differences to be computed. The usual convention is that ϕ is zero at infinity, but note that it's not always possible.

Any path can be taken. Consider the difference between two paths:

$$-\int_a^b d\underline{r}_1\cdot\underline{\nabla}\phi + \int_a^b d\underline{r}_2\cdot\underline{\nabla}\phi$$

This integral is just an integral around a loop that is closed:

$$\oint_C d\underline{r}\cdot\underline{E} = \int_S \underline{\nabla}\times\underline{E}\cdot d\underline{S} = 0$$

Since we are taking the static case where $-\partial_t \underline{B} = 0$

6.2 Poisson's equation for ϕ

Since $\underline{E} = -\underline{\nabla}\phi$ and $\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0}$, and

$$\underline{\nabla}\cdot(-\underline{\nabla}\phi) = -\underline{\nabla}^2\phi$$

Hence ϕ obeys

$$\underline{\nabla}^2\phi = -\frac{\rho}{\epsilon_0}$$

This is Poisson's Equation

The solution is essentially via the Green's function technique. For a point charge $\delta q'$ at $\underline{r}'$ the charge density is

$$\rho(\underline{r}) = \delta q'\,\delta(\underline{r}-\underline{r}')$$

which has a scalar potential. (Coulomb law)

$$\phi(\underline{r}) = \frac{\delta q'}{4\pi\epsilon_0|\underline{r}-\underline{r}'|}$$

Solve this informally by dividing up a general charge distribution $\rho(\underline{r}')$ into small cells of size dV, containing a charge $\delta q_i = \rho(\underline{r}')dV$.

We add those up

$$\phi(\underline{r}) = \sum_i \frac{\delta q_i}{4\pi\epsilon_0|\underline{r}-\underline{r}_i|}$$

Taking the cell size to zero, the solution to Poisson's equation is

$$\phi(\underline{r}) = \int_{V'} \frac{\rho(\underline{r}')}{4\pi\epsilon_0|\underline{r}-\underline{r}'|}\,dV'$$

This only holds for static case (V' indicates integral is over $\underline{r}'$)

6.2.1 Solutions to Laplace's equation are unique

Dirichlet and Neumann boundary conditions

A common situation in electrostatics is to find the solutions to the electrostatic potential in a vacuum, with some boundary conditions that specify either the potential or the normal component of the electric field on some bounding surface.

Dirichlet b.c. specify the potential on the boundary, whereas Neumann b.c. specify $\underline{\nabla}\phi \cdot \hat{d\underline{s}}$, where $\hat{d\underline{s}}$ is a unit vector. The problem is determined by one or the other b.c., not both.

The solution to $\underline{\nabla}^2\phi = 0$ subject to either Dirichlet or Neumann b.c. is unique.

Proof: Assume we have solutions $\phi_1(\underline{r})$ and $\phi_2(\underline{r})$. We will show that they have to be the same.

Let $\psi = \phi_1 - \phi_2$, we will prove that $\psi = 0$ (Dirichlet), or is (an irrelevant) constant at most (Neumann). Consider

$$\int_V \underline{\nabla}\cdot(\psi\underline{\nabla}\psi)\,dV = \int_V (\underline{\nabla}\psi\cdot\underline{\nabla}\psi + \psi\underline{\nabla}^2\psi)\,dV$$

But $\underline{\nabla}^2\psi = 0$, so using divergence theorem

$$\oint_S (\psi\underline{\nabla}\psi)\cdot d\underline{s} = \int_V (\underline{\nabla}\psi\cdot\underline{\nabla}\psi)\,dV$$

But the LHS is 0, since either $\phi_1 = \phi_2$ or $\underline{\nabla}\phi_1\cdot d\underline{s} = \underline{\nabla}\phi_2\cdot d\underline{s}$

i.e. either $\psi = 0$ or $\underline{\nabla}\psi\cdot d\underline{s} = 0$ on the boundary

$$\int_V (\underline{\nabla}\psi\cdot\underline{\nabla}\psi)\,dV = 0$$

which requires $|\underline{\nabla}\psi|^2 = 0$, so $\underline{\nabla}\psi = 0$ everywhere, so (up to an irrelevant additive constant in the Neumann case),

$$\phi_1 = \phi_2$$

such that all solutions are the same.

6.3 Magnetostatics

6.3.1 Magneto Vector Potential

Magnetic field in the static case, $\underline{\nabla}\cdot\underline{B}=0$ and $\underline{\nabla}\times\underline{B}=\mu_0\underline{J}$, nothing depends on t.

Maths: divergence-free fields

If $\underline{F}(\underline{r})$ is a vector field that satisfies $\underline{\nabla}\cdot\underline{F}=0$, then there is a vector field $\underline{G}$ such that

$$\underline{F}=\underline{\nabla}\times\underline{G}$$

The converse is also true.

Because $\underline{\nabla}\cdot\underline{B}=0$, there is a field, the Magnetic vector Potential, A, s.t.

$$\underline{B}=\underline{\nabla}\times\underline{A}$$

6.4 Gauge transformation

The magnetic vector potential is not unique. This is more important and more subtle than the non-uniqueness of the electrostatic potential. We can add the gradient of any scalar field $\psi(r)$ to $\underline{A}$ and it will not change $\underline{B}$. This is an example of gauge transformation:

$$\underline{A}'=\underline{A}+\underline{\nabla}\psi$$

$$\underline{\nabla}\times\underline{A}'=\underline{\nabla}\times\underline{A}+\underline{\nabla}\times\underline{\nabla}\psi=\underline{\nabla}\times A$$

and $\underline{\nabla}\times\underline{\nabla}\psi\equiv 0$, so $\underline{B}'=\underline{B}$.

6.4.1 Gauge Condition

If we want to make $\underline{A}$ unique, we need to impose an additional condition (like $\psi\to 0$ at infinity). This is called a gauge condition.

Many choices are possible, but two are common.

1: Coulomb gauge (magnetostatics)

$$\underline{\nabla}\cdot\underline{A}=0$$

2: Lorentz gauge (time-dependent cases)

$$\underline{\nabla}\cdot\underline{A}+\frac{1}{c^2}\partial_t\phi=0$$

In magnetostatics

$$\underline{\nabla}\times\underline{B}=\underline{\nabla}\times(\underline{\nabla}\times\underline{A})=\underline{\nabla}(\underline{\nabla}\cdot\underline{A})-\underline{\nabla}^2\underline{A}=\mu_0\underline{J}$$

so in the Coulomb gauge

$$\underline{\nabla}^2\underline{A}=-\mu_0\underline{J}$$

We can solve this component by component in the same way as for the electrostatic potential:

$$A(r)=\int_{V'}\frac{\mu_0 J(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|}dV'$$

We derived the magnetic vector potential A assuming the Coulomb gauge. Let us check that the solution satisfies $\underline{\nabla}\cdot\underline{A}=0$

$$\underline{\nabla}\cdot\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\,\underline{\nabla}\cdot\int_{V'}\frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\,dV'$$

Note that $\underline{\nabla}$ differentiates with respect to $\underline{r}$, not $\underline{r}'$ and the only part of the RHS of the equation that depends on $\underline{r}$ is $|\underline{r}-\underline{r}'|^{-1}$

$$\underline{\nabla}\cdot\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_{V'}\underline{J}(\underline{r}')\cdot\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)dV'$$

Since $\quad \underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) = -\underline{\nabla}'\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)$

$$\underline{\nabla}\cdot\underline{A}(\underline{r}) = -\frac{\mu_0}{4\pi}\int_{V'}\underline{J}(\underline{r}')\cdot\underline{\nabla}'\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)dV'$$

Maths: Divergence of scalar times vector field

For any scalar field $\psi(\underline{r})$ and vector field $\underline{F}(\underline{r})$

$$\underline{\nabla}\cdot(\psi\underline{F}) = \psi\,\underline{\nabla}\cdot\underline{F} + \underline{F}\cdot\underline{\nabla}\psi$$

Writing $\psi = \frac{1}{|\underline{r}-\underline{r}'|}$,

$$\underline{J}\cdot\underline{\nabla}'\psi = \underline{\nabla}'\cdot(\psi\underline{J}) - \psi\,\underline{\nabla}'\cdot\underline{J}$$

$$\underline{\nabla}\cdot\underline{A}(\underline{r}) = \frac{-\mu_0}{4\pi}\int_{V'}\left[\underline{\nabla}'\cdot\left(\frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\right) - \underline{\nabla}'\cdot\underline{J}(\underline{r}')\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\right]dV'$$

But $\underline{\nabla}'\cdot\underline{J} = -\partial_t\rho(\underline{r}') = 0$ from the continuity equation, and we are considering static solutions. Hence using the Divergence theorem

$$\underline{\nabla}\cdot\underline{A}(\underline{r}) = -\frac{\mu_0}{4\pi}\int_{V'}\underline{\nabla}'\cdot\left(\frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\right)dV'$$

$$= -\frac{\mu_0}{4\pi}\oint_{\underline{S}'}\frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\cdot d\underline{S}' = 0$$

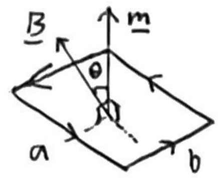
If we take the boundary to infinity, then, provided that the current density is localised and therefore vanishes on the boundary, the surface integral vanishes. Solution satisfies Coulomb gauge condition.

6.5 Magnetic Dipoles

A rectangular loop of sides a and b, carrying a current I has a magnetic dipole

$$|\underline{m}| = abI$$

in the direction perpendicular to the loop



Magnetic dipole formed by a current loop

The force on an electron is $e(\underline{v} \times \underline{B})$, so the force per unit length on a current I is

$$\underline{F} = nq(\underline{V} \times \underline{B}) = \underline{I} \times \underline{B}$$

where n is the number of charges per unit length. If $\underline{B}$ is uniform, and in the direction shown, there is no net force on the a or b arms, as the forces are equal and opposite on the opposite arms (because $\underline{I}$ reverses direction). So there is no net force if $\underline{B}$ is uniform. There will be a net force if $\underline{B}$ varies with position (there is also a torque)

The total force on the current loop is

$$\underline{F} = \oint \underline{I} \times \underline{B} \, d\underline{l}$$

and the magnetic field is, to first order

$$\underline{B}(x,y,z) = \underline{B}_0 + (\underline{r} \cdot \underline{\nabla})\underline{B}$$

where B_0, is the corner of the loop, is the field at the origin.

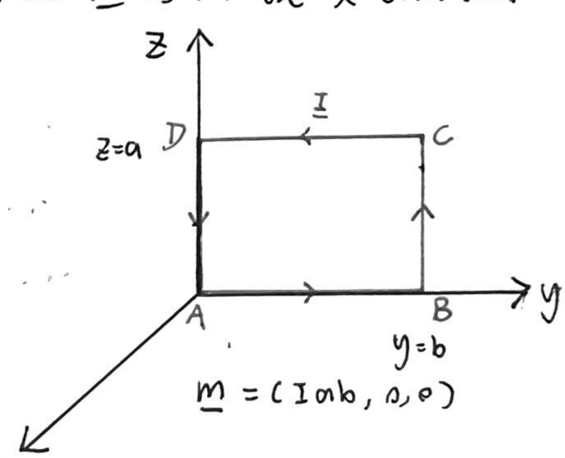
Let's put the loop in the y-z plane, so $\underline{m}$ is in the x direction

The magnetic field in the AB section is

$$\underline{B}(0,y,0) = \underline{B}_0 + y\,\partial_y \underline{B} \qquad (6.43)$$

where all derivatives are evaluated at the origin, so are constants. Consider section CD:

$$\underline{B}(0,y,a) = \underline{B}_0 + y\,\partial_y \underline{B} + a\,\partial_z \underline{B} \qquad (6.44)$$



$\underline{m} = (Iab, 0, 0)$

Since current flows in opposite directions in AB and CD, we only consider the last term of (6.44). This gives a net force

$$F_{AB+CD} = b\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -I & 0 \\ a\partial_z B_x & a\partial_z B_y & a\partial_z B_z \end{vmatrix} = abI\begin{pmatrix} -\partial_z B_z \\ 0 \\ \partial_z B_x \end{pmatrix}$$

$$F_{BC+DA} = a\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & I \\ b\partial_y B_x & b\partial_y & b\partial_y B_z \end{vmatrix} = abI\begin{pmatrix} -\partial_y B_y \\ \partial_y B_x \\ 0 \end{pmatrix}$$

The net force is ~~therefore~~

$$\underline{F} = abI \begin{pmatrix} -\partial_y B_y - \partial_z B_z \\ \partial_y B_x \\ \partial_z B_x \end{pmatrix} = abI \begin{pmatrix} \partial_x B_x \\ \partial_y B_x \\ \partial_z B_x \end{pmatrix} = \underline{\nabla}(I ab B_x)$$

where we used $\underline{\nabla} \cdot \underline{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z = 0$ to simply the first term.

Now the magnetic moment $\underline{m} = (abI, 0, 0)$ is constant and in the x direction, so we see that to obtain the force, we project $\underline{B}$ onto the direction of $\underline{m}$: $abI\, B_x = \underline{m} \cdot \underline{B}$

$$\underline{F} = \underline{\nabla}(\underline{m} \cdot \underline{B})$$

In other words, there is a potential ($\underline{F} = -\underline{\nabla} U$) associated with a magnetic dipole moment $\underline{m}$ in a magnetic field $\underline{B}$ given by

$$U = -\underline{m} \cdot \underline{B}$$

6.6 The magnetic field of a magnetic dipole

Lets work out the magnetic field of a static current loop C of dimension $\simeq d$ near the origin.

The magnetic vector potential is given in terms of the current density $\underline{J}$

$$\underline{A}(\underline{r}) = \int_{V'} \frac{\mu_0 \underline{J}(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|} dV' = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\underline{r}'}{|\underline{r}-\underline{r}'|}$$

where r' now runs around the loop carrying a current I

If we look from afar, $|r| \gg d$, then we can expand the inverse distance in a Taylor expansion:

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r} + \underline{r}' \cdot \underline{\nabla}'\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) + \cdots$$

Consider the x' derivative

$$\frac{\partial}{\partial x'}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) = \frac{\partial\left[\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}\right]}{\partial x'} = \frac{x-x'}{|\underline{r}-\underline{r}'|^3}$$

In the Taylor expansion, we evaluate this at $r'=0$

$$\underline{\nabla}'\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) = \frac{\underline{r}}{r^3}$$

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r} + \frac{\underline{r}'\cdot\underline{r}}{r^3} + \cdots$$

when we put this in the equation for $\underline{A}$, the $\frac{1}{r}$ term integrates to zero:

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\underline{r}'\cdot\underline{r}}{r^3} d\underline{r}'$$

Now we use a trick to be able to use Stokes' theorem on this. We need a vector dotted with $d\underline{r}'$ to use Stokes' theorem, so let us take the scalar product of the integral above with an arbitrary constant vector $\underline{u}$:

$$\underline{A}(\underline{r})\cdot\underline{u} = \frac{\mu_0 I}{4\pi r^3} \oint_C [(\underline{r}'\cdot\underline{r})\underline{u}]\cdot d\underline{r}'$$

$$= \frac{\mu_0 I}{4\pi r^3} \int_{S'} [\underline{\nabla}' \times (\underline{r}'\cdot\underline{r})\underline{u}]\cdot d\underline{S}'$$

$$= -\frac{\mu_0 I}{4\pi r^3} \int_{S'} [\underline{u} \times \underline{\nabla}'(\underline{r}'\cdot\underline{r})]\cdot d\underline{S}'$$

The last line follows the vector identity:

$$\underline{\nabla}\times(\psi\underline{F}) = \psi\underline{\nabla}\times\underline{F} - \underline{F}\times\underline{\nabla}\psi$$

where $\underline{\nabla}\times\underline{u}$ vanishes since $\underline{u}$ is constant

Now consider
$$\frac{\partial(\underline{r}\cdot\underline{r}')}{\partial x'} = \frac{\partial(xx'+yy'+zz')}{\partial x'} = x$$

so $\underline{\nabla}'(\underline{r}\cdot\underline{r}') = \underline{r}$ and
$$\underline{A}(\underline{r})\cdot\underline{u} = -\frac{\mu_0 I}{4\pi r^3}\int_{S'}(\underline{u}\times\underline{r})\cdot d\underline{s}'$$
$$= -\frac{\mu_0 I}{4\pi r^3}\int_{S'}\underline{u}\cdot(\underline{r}\times d\underline{s}')$$

where we used the scalar triple vector identity:
$$(\underline{a}\times\underline{b})\cdot\underline{c} = (\underline{b}\times\underline{c})\cdot\underline{a} = (\underline{c}\times\underline{a})\cdot\underline{b}$$
Taking the $\underline{u}$ outside the integral, and noting that it's arbitrary:
$$\underline{A}(\underline{r}) = -\frac{\mu_0 I}{4\pi r^3}\underline{r}\times\int_{S'}d\underline{s}' = -\frac{\mu_0 I}{4\pi r^3}\underline{r}\times\underline{S}$$

$\underline{S}$ points out of the loop, and the magnetic moment is $\underline{m} = I\underline{S}$, so
$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi r^3}\underline{m}\times\underline{r}$$

The magnetic field is $\underline{B} = \underline{\nabla}\times\underline{A}$ and using:
$$\underline{\nabla}\times(\psi\underline{F}) = \psi\underline{\nabla}\times\underline{F} - \underline{F}\times\underline{\nabla}\psi$$
$$\underline{\nabla}\times(\underline{F}\times\underline{G}) = \underline{F}(\underline{\nabla}\cdot\underline{G}) - \underline{G}(\underline{\nabla}\cdot\underline{F}) - (\underline{F}\cdot\underline{\nabla})\underline{G} + (\underline{G}\cdot\underline{\nabla})\underline{F}$$
with the identities, and noting that $\underline{m}$ is a constant vector (static loop),
$$\underline{\nabla}\times\left(\frac{1}{r^3}\underline{m}\times\underline{r}\right) = \frac{1}{r^3}\underline{\nabla}\times(\underline{m}\times\underline{r}) - (\underline{m}\times\underline{r})\times\underline{\nabla}\left(\frac{1}{r^3}\right)$$
$$= \frac{1}{r^3}\left[\underline{m}(\underline{\nabla}\cdot\underline{r}) - \cancel{\underline{r}(\underline{\nabla}\cdot\underline{m})} - (\underline{m}\cdot\underline{\nabla})\underline{r} + \cancel{(\underline{r}\cdot\underline{\nabla})\underline{m}}\right] + (\underline{m}\times\underline{r})\times\frac{3\underline{r}}{r^5}$$

where $\underline{\nabla}\left(\frac{1}{r^3}\right) = -\frac{3\underline{r}}{r^5}$, $\underline{\nabla}\cdot\underline{r} = 3$, and $(\underline{m}\cdot\underline{\nabla})\underline{r} = \underline{m}$, and using ~~$(\underline{m}\times\underline{r})\times\underline{r}$~~
$(\underline{m}\times\underline{r})\times\underline{r} = (\underline{m}\cdot\underline{r})\underline{r} - r^2\underline{m}$, we find
$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\underline{m}\cdot\hat{\underline{r}})\hat{\underline{r}} - \underline{m}}{r^3}\right]$$

This has exactly the same form as the electric field from a dipole of charges, but note that the correspondence is only good at large distance.

Ch7 Time-dependent electromagnetism

Let's consider the general case when $\underline{A}(\underline{r},t)$ and $\phi(\underline{r},t)$ are time-dependent

$\underline{\nabla}\cdot\underline{B}=0$, so we still have

$$\underline{B}(\underline{r},t)=\underline{\nabla}\times\underline{A}(\underline{r},t)$$

For the scalar potential, we note that

$$\underline{\nabla}\times\underline{E}=-\partial_t\underline{B}=-\partial_t(\underline{\nabla}\times\underline{A})$$

$$\underline{\nabla}\times(\partial_t\underline{A}+\underline{E})=0$$

$$\partial_t\underline{A}+\underline{E}=-\underline{\nabla}\phi$$

$$\underline{E}=-\underline{\nabla}\phi-\partial_t\underline{A}$$

7.1 Wave equations for ϕ and $\underline{A}$

We have only used two of Maxwell's equations so far.

$$\underline{\nabla}\times\underline{B}=\mu_0\underline{J}+\frac{1}{c^2}\partial_t\underline{E}$$

Using a vector identity, and substituting for $\underline{E}$:

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{A})=\underline{\nabla}(\underline{\nabla}\cdot\underline{A})-\underline{\nabla}^2\underline{A}=\mu_0\underline{J}+\frac{1}{c^2}\partial_t(-\underline{\nabla}\phi-\partial_t\underline{A})$$

$$\underline{\nabla}^2\underline{A}-\frac{1}{c^2}\partial^2_{tt}\underline{A}=-\mu_0\underline{J}+\underline{\nabla}(\underline{\nabla}\cdot\underline{A}+\frac{1}{c^2}\partial_t\phi)$$

This is messy but can be simplified by choosing the Lorenz Gauge:

$$\underline{\nabla}\cdot\underline{A}=-\frac{1}{c^2}\partial_t\phi$$

and the wave equation for $\underline{A}$ in this gauge is

$$\underline{\nabla}^2\underline{A}-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=\mu_0\underline{J}$$

with J as the source. For the scalar potential, we use

$$\underline{\nabla}\cdot\underline{E}=\frac{\rho}{\epsilon_0}=-\underline{\nabla}\cdot(\underline{\nabla}\phi+\partial_t\underline{A})=-\underline{\nabla}^2\phi-\partial_t(\underline{\nabla}\cdot\underline{A})$$

and in the Lorenz gauge,

$$\partial_t(\underline{\nabla}\cdot A)=-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\frac{\rho}{\epsilon_0}=-\underline{\nabla}^2\phi+\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

And we have another wave equation for ϕ

$$\underline{\nabla}^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho}{\epsilon_0}$$

with ρ as the source

We have two driven wave equations, for $\underline{A}$ and ϕ, with $\underline{J}$ and ρ as sources.

Gauge transformations

If we change $\underline{A} \to \underline{A} + \underline{\nabla}\psi$

then $\underline{B}$ is unchange, as required.

Since $\underline{\nabla}\cdot\underline{A} \to \underline{\nabla}\cdot\underline{A} + \underline{\nabla}^2\psi$, we are free to choose $\underline{\nabla}\cdot\underline{A}$ to be whatever we like, for example to satisfy the Lorentz condition, by choosing ψ suitably. The Lorentz gauge has no special meaning. Many different $\underline{A}$ fields describe exactly the same physics and it's a techical choice to make the equation simpler.

What happens to $\underline{E}$ if we make the above gauge transformation? In the time-dependent case, $\underline{E}$ depends on $\underline{A}$ through $\underline{E} = -\underline{\nabla}\phi - \partial_t\underline{A}$.

In order to keep $\underline{E}$ unchanged, we need to change ϕ:

$$\phi \to \phi - \partial_t \psi$$

So $$\underline{E} \to -\underline{\nabla}(\phi - \partial_t\psi) - \partial_t(\underline{A} + \underline{\nabla}\psi) = -\underline{\nabla}\phi - \partial_t\underline{A}$$

is unchanged.

7.2 Solving the wave equations for ϕ and A

Reminder for Green's function technique

If we have a linear differential equation

$$\hat{L}[y(x)] = f(x)$$

where $\hat{L}$ is a linear operator and f is some source function, then if we can find a solution to the Green's function equation, for a point source at x',

$$\hat{L}[G(x,x')] = \delta(x-x')$$

then the solution for a general source $f(x)$ is

$$y(x) = \int G(x,x') f(x') dx'$$

which arises since

$$f(x) = \int \delta(x-x') f(x') dx'$$

from the sifting property of Dirac delta functions

Let's ~~be~~ do this in four dimensions. We replace the time and spatially dependent charge density by a point charge at position r' at time t'

$$\rho(\underline{r},t) \rightarrow \delta(\underline{r}-\underline{r}')\delta(t-t')$$

The solution to the equation is the Green's function $G(\underline{r},t;\underline{r}',t)$, which is the response at $\underline{r}, t$ to this point charge. It satisfies

$$\underline{\nabla}^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -\frac{1}{\epsilon_0}\delta(\underline{r}-\underline{r}')\delta(t-t') \qquad (7.19)$$

Let's remove the time derivatives by taking Fourier transform w.r.t. t:

$$\tilde{G}(\underline{r},\omega;\underline{r}',t') \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\underline{r},t;\underline{r}',t') e^{i\omega t} dt$$

The second time derivative transforms simply to

$$\frac{\partial^2 G}{\partial t^2} \rightarrow -\omega^2 \tilde{G}$$

The RHS of 7.19 transforms using

$$\int_{-\infty}^{\infty} \delta(t-t') e^{i\omega t} dt = e^{i\omega t'}$$

Therefore $\tilde{G}$ satisfies

$$\underline{\nabla}^2 \tilde{G} + \frac{\omega^2}{c^2}\tilde{G} = -\frac{1}{\sqrt{2\pi}\,\epsilon_0}\delta(r-r') e^{i\omega t'}$$

Letting $U(\underline{r},\omega;r',t') = \sqrt{2\pi}\,\tilde{G}\,e^{-i\omega t'}$ we find

$$\underline{\nabla}^2 U + \frac{\omega^2}{c^2} U = -\frac{1}{\epsilon_0}\delta(\underline{r}-\underline{r}')$$

By symmetry, U can depend on $\underline{r}$ and $\underline{r}'$ through $R \equiv |\underline{R}| \equiv |\underline{r}-\underline{r}'|$ so it makes sense to use spherical polars. In this case $\underline{\nabla}^2$ simplifies to

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial U}{\partial R}\right) + \frac{\omega^2}{c^2}U = -\frac{1}{\epsilon_0}\delta(\underline{R})$$

where we have used the spherical symmetry to simplify $\underline{\nabla}^2$ by including only R derivatives

To solve this, we simplify it by making a substitution

$$V = RU$$

which satisfies

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{V}{R}\right)\right) + \frac{\omega^2}{c^2}\frac{V}{R} = -\frac{1}{\epsilon_0}\delta(\underline{R})$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R} - V\right) + \frac{\omega^2}{c^2}\frac{V}{R} = -\frac{1}{\epsilon_0}\delta(\underline{R})$$

$$\frac{\partial^2 V}{\partial R^2} + \frac{\omega^2}{c^2}V = -\frac{R}{\epsilon_0}\delta(\underline{R})$$

This is now a standard harmonic equation with delta function source. We know the general solution (which is a function of R and t only). Except at $\underline{R}=0$, the equation is harmonic. Fixing ω for now, we see that

$$V(R,\omega) = Ae^{i\omega R/c} + Be^{-i\omega R/c}$$

where A, B are constants. Hence

$$\tilde{G}(\underline{r},\omega;\underline{r}',t') = \frac{1}{\sqrt{2\pi}\,R}\left(Ae^{i\omega R/c} + Be^{-i\omega R/c}\right)e^{i\omega t'}$$

Now we invert the Fourier Transform to find $\underline{G}$:

$$G(\underline{r},t;\underline{r}',t') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,R}\left(Ae^{i\omega R/c} + Be^{-i\omega R/c}\right)e^{i\omega t'}e^{-i\omega t}\,d\omega$$

use $\int_{-\infty}^{\infty} e^{i\omega T}d\omega = 2\pi\delta(T)$

the other term gives $G = \frac{1}{R}\left\{A\delta(t'-[t-\frac{R}{c}]) + B\delta(t'-[t+\frac{R}{c}])\right\}$

we notice that the B term obtains contributions only from $t' = t+\frac{R}{c}$ i.e. the future, so it is unphysical and we set $B=0$

7.3 Retarded potentials and retarded time

Near the point charge, when $R \to 0$, the potential must look like the stationary solution, which means $A = \frac{1}{4\pi\epsilon_0}$ (remember it's a unit charge), and

$$G(r,t;r',t') = \frac{1}{4\pi\epsilon_0 |\underline{r}-\underline{r}'|}\delta\left(t' - \left[t - \frac{|\underline{r}-\underline{r}'|}{c}\right]\right)$$

This is called the Retarded Green's Function, because it reflects the causality of the disturbance: the field at a point is determined by the position of charges at a time in the past (retarded time) such that their influence just reaches the point at time t. The retarded time is:

$$t_{ret} = t - \frac{|\underline{r}-\underline{r}'|}{c}$$

Adding contributions from charge densities everywhere, we integrate the Green's function over $\underline{r}'$ and t', multiplied by $\rho(\underline{r}',t')$:

$$\phi(\underline{r},t) = \int \rho(r',t') \frac{1}{4\pi\epsilon_0 |\underline{r}-\underline{r}'|}\delta\left(t' - \left[t - \frac{|\underline{r}-\underline{r}'|}{c}\right]\right) dt' dr'$$

The t' integral is easy, as it has a delta function, replacing t' by the retarded time in ρ, and giving

$$\phi(\underline{r},t) = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho(r', t - |\underline{r}-\underline{r}'|/c)}{|\underline{r}-\underline{r}'|} dr'$$

By the same reasoning, we find the retarded solution for $\underline{A}$ is

$$\underline{A}(\underline{r},t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\underline{J}(\underline{r}', t - |\underline{r}-\underline{r}'|/c)}{|\underline{r}-\underline{r}'|} dr'$$

Once these have been computed, we find $\underline{E}$ and $\underline{B}$ from

$$\underline{E} = -\underline{\nabla}\phi - \partial_t \underline{A}$$

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

3.0 EM Fields in Matter (EM Part 2)

3.1 Formalism

There are a few types of charge

- $\rho_f (K_f)$ are the body (and surface) free charge densities. In a dielectric these are both zero
- $\rho_p (K_p)$ are the body (and surface) polarisation charge densities. These are due to bound charge being displaced by an electric field

We also consider a few types of current

- $\underline{J}_c (\underline{K}_c)$ are the body (and surface) conduction current densities. In a dielectric these are both zero
- $\underline{J}_p$ is the body polarisation current density, due to time variation of the polarisation charge
- $\underline{J}_m (\underline{K}_m)$ are the body (and surface) magnetisation current densities due to ~~bond~~ bound magnetic dipoles

The surface charge density has units $[K] = C/m^2$, volume charge density $[\rho] = C/m^3$. The surface current density has units $[\underline{K}] = A/m$, and body current density $[\underline{J}] = A/m^2$

Maxwell's laws are still gross, so we will later try to tidy them up.

$$\underline{\nabla} \cdot \underline{E} = (\rho_f + \rho_p)/\varepsilon_0$$

$$\underline{\nabla} \cdot \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\partial_t \underline{B}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 (\underline{J}_c + \underline{J}_m + \underline{J}_p) + \varepsilon_0 \mu_0 \partial_t \underline{E}$$

Polarisation:

- Materials can respond to the application of electric field by changing their electric dipole moment – this response changes the $\underline{E}$ field in the material, and we have to adjust Maxwell equations to make our life easier.
- Introduce the polarisation field $\underline{P}$ which is the dipole moment per unit volume. It can be written $\underline{P} = N\underline{p} = Nq\underline{d}$ where these terms are the atomic number density N and atomic dipole moment $\underline{p}$ which equals charge x displacement. Writing the polarisation field as a function of the applied field: $\underline{P} = \chi_e \varepsilon_0 \underline{E}$

For now we make no assumptions about χ_e, which is generally a function of $\underline{E}$ and is an anisotropic tensor

↳ Now we express the surface polarisation charge density in terms of polarisation. In our dielectric, the free charge is zero, and the bound charge is conserved at zero. Write total charges $Q_{surface} + Q_{volume} = 0$, and notice that these terms can be written as the integral over surface and volume charge density : $\oiint_S K_p \, dS + \iiint_V \rho_p \, dV = 0$

↳ Consider an infinitesimal volume in the material. As the surface charge is caused by a displacement $\underline{d}$ of charges q with volume density N, write the infinitesimal surface charge as the product of charge density Nq with the volume of the layer sticking out of the surface $dq_{sp} \equiv K_p dS = qN\underline{d} \cdot d\underline{S}$

$K_p dS = \underline{P} \cdot d\underline{S}$

Now write the above integral in terms of the polarisation field and use the divergence theorem

$$\oiint_S \underline{P} \cdot d\underline{S} + \iiint \rho_p dV = 0 \rightarrow \iiint (\nabla \cdot \underline{P} + \rho_p) dV = 0$$

which tells us that $\rho_p = -\nabla \cdot \underline{P}$

Now define the D-field version of the electric field to make Gauss' law easier

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$$

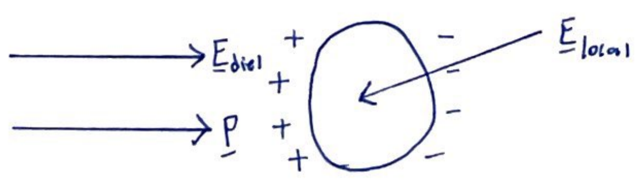
and then get this into Gauss' law

$$\varepsilon_0 (\nabla \cdot \underline{E}) = \rho_f + \rho_p = \rho_f - \nabla \cdot \underline{P} \rightarrow \nabla \cdot (\varepsilon_0 \underline{E} + \underline{P}) = \rho_f = \nabla \cdot \underline{D}$$

3.2 Dielectrics

Dielectrics are insulators : they have no free charge. They are also not magnetic. When an dielectric is placed in a capacitor, the E field changes but the $\underline{D}$ field remains constant. The energy stored changes too as $W_E = \frac{1}{2}\varepsilon E^2 = \frac{1}{2} ED$

Clausius - Mossotti Equation : This equation relates the atomic dipole moment to the local electric field by a constant called the atomic polarisability $\underline{p} = \alpha E_{local}$. We can use this constant to now estimate the relative permittivity, but have to be careful to note that the electric field the atom feels $\underline{E}_{local} \neq \underline{E}_{diel}$ in the rest of the dielectric



The $\underline{E}_{local}$ is the field at the center of a spherical cavity in the dielectric (formed by removing the atom). There is a modification because on the sides of the cavity the polarisation field acts to cancel some field.

By integrating the charge on the inside surface of the spherical cavity, the correction to the electric field is found to be $\underline{P}/3\varepsilon_0 \Rightarrow \underline{E}_{local} = \underline{E}_{diel} + \underline{P}/3\varepsilon_0$

Now $\underline{P} = N\underline{p} = N\alpha \underline{E}_{local} = N\alpha(\underline{E}_{diel} + \underline{P}/3\varepsilon_0) \rightarrow$ rearrange to give ~~$N\alpha E_{diel}$~~

$N\alpha \underline{E}_{diel} = \underline{P}(1 - N\alpha/3\varepsilon_0)$.

Using the definition of polarisation, $\underline{P} = \chi_e \varepsilon_0 \underline{E}_{diel}$, we can show

$$\chi_e \varepsilon_0 (1 - N\alpha/3\varepsilon_0) = N\alpha$$

$$\varepsilon_r = 1 + \chi_e = \frac{1 + 2N\alpha/3\varepsilon_0}{1 - N\alpha/3\varepsilon_0}$$

EM Part 2

15 Dielectrics

15.1 Polarization

Apply electric field to dielectric → atoms become aligned electric dipoles

Define $\underline{P}$ = polarization = atomic (or molecular) dipole moment / volume

$= N\underline{p}$, where N = number of atoms / volume

$\underline{p}$ = atomic (or molecular) dipole moment $= q\underline{d}$

$\underline{d}$ = vector displacement from − to +

Write $\underline{P} = \chi_e \varepsilon_0 \underline{E}$, where χ_e = electric susceptibility

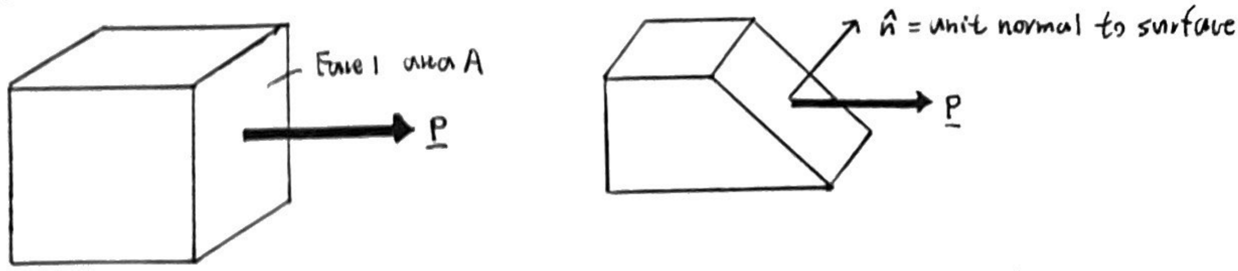
For many materials:

- $\underline{P}$ is parallel to $\underline{E}$ → material is isotropic and χ_e is a scalar
  ↳ Anisotropic materials are easier to polarize in some directions than others. In such materials $\underline{P}$ is not parallel to $\underline{E}$ and χ_e is a tensor.
- χ_e is independent of $\underline{E}$ → material is linear
- χ_e is uniform → material is homogeneous

HIL = homogeneous, isotropic, linear

15.2 Charge and current densities

Consider a small block of dielectric in an electric field (left)

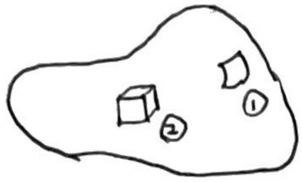


The number of dipoles sticking out of face 1 of the block is NAd as the volume of a layer one dipole thick is Ad. ⇒ the charge on the surface $= qNdA = PA$

On the right the same number of dipoles are sticking out of the surface (same face charge) but the area is now $A/\cos\theta$

Therefore ρ_{sp} = polarization surface charge density in $C\,m^{-2} = \dfrac{PA}{A/\cos\theta} = \underline{\hat{n}} \cdot \underline{P}$

Piece of dielectric : total charge = 0



① Charge on surface element $= \underline{P} \cdot d\underline{S}$

② Charge in volume element $= \rho_p dV$ (where ρ_p = polarization charge density)

$Q_{inside} + Q_{surf}$ = total charge = 0
① ②

① $= \int \rho_p dV$ ② $= \oint \underline{P} \cdot d\underline{S} = \int \nabla \underline{P} \cdot dV$

Therefore : $\int (\rho_p + \nabla \underline{P}) dV = 0 \rightarrow \rho_p = -\nabla \underline{P}$ ($= 0$ if $\underline{P}$ is uniform)

In general it's useful to make distinction between two types of charge density

(1) ρ_f, ρ_{sf} : free charge density and surface charge density (both zero in dielectrics)

(2) ρ_p, ρ_{sp} : polarisation charge density and surface charge density (from bound charges)

If $\underline{P}$ is time varying :

$$\frac{\partial \underline{P}}{\partial t} = N \frac{\partial \underline{p}}{\partial t} = Nq \frac{\partial \underline{d}}{\partial t} = Nq \underline{v}$$

This is the current density in Am^{-2} due to movement of the bound charges, i.e. $\underline{J}_p$ = polarization current density $= \frac{\partial \underline{P}}{\partial t}$

15.3 D and ε_r

$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \varepsilon_0 \nabla \underline{E} = \rho_f + \rho_p = \rho_f - \nabla \cdot \underline{P} \rightarrow \nabla(\varepsilon_0 \underline{E} + \underline{P}) = \rho_f$

Define : $\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$ = "electric displacement"

Remember E is the electric field in the dielectric

$\rightarrow \nabla \cdot \underline{D} = \rho_f$ or $\oint \underline{D} \cdot d\underline{S} = Q_f^{enc}$

$\underline{P} = \chi_e \varepsilon_0 \underline{E} \rightarrow \underline{D} = (1 + \chi_e) \varepsilon_0 \underline{E} = \varepsilon \underline{E}$

where $\varepsilon = \varepsilon_0 \varepsilon_r$

where $\varepsilon_r = 1 + \chi_e$ = relative permittivity

15.4 Maxwell's equation in HIL dielectrics

- $\nabla \cdot \underline{D} = \rho_f$ But in dielectrics $\rho_f = 0$ and $\underline{D} = \varepsilon_0 \varepsilon_r \underline{E} \rightarrow \underline{\nabla} \cdot \underline{E} = 0$
- $\nabla \cdot \underline{B} = 0$
- $\nabla \times \underline{E} = -\dfrac{\partial B}{\partial t}$
- $\underbrace{\nabla \times \underline{H}}_{①} = \underbrace{\underline{J}_c}_{②} + \underbrace{\dfrac{\partial D}{\partial t}}_{③}$

① : in most dielectrics $\underline{H} = \dfrac{B}{\mu_0}$

② = conduction density $= 0$ in dielectric

③ $= \varepsilon_0 \varepsilon_r \dfrac{\partial E}{\partial t}$

$\rightarrow \nabla \times \underline{B} = \mu_0 \varepsilon_0 \varepsilon_r \dfrac{\partial E}{\partial t}$

The equations are identical to MEs in vacuum, but with ε_0 replaced with $\varepsilon_0 \varepsilon_r$.

15.5 Clausius - Mossotti equation

An atom / molecule becomes polarized when an $\underline{E}$ field is applied.

$\underline{p} = \alpha \underline{E}$, α = the polarizability of the atom / molecule.

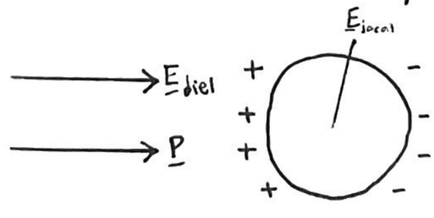
The C-M equation relates α to the relative permittivity ε_r.

place a dielectric in an applied vacuum electric field $\underline{E}_{vac}$

$\underline{E}_{diel}$ = electric field in dielectric $\neq \underline{E}_{vac}$ i.e. vacuum field modified by the dielectric

$\underline{E}_{local}$ = electric field experienced by any atom in the dielectric $\neq \underline{E}_{diel}$ because there is an additional modification due to the adjacent atomic dipoles

Assume $\underline{E}_{local}$ at atom is $\underline{E}$ at the center of the spherical cavity formed when the atom is removed.



$\underline{E}_{local} = \underline{E}_{diel} + \dfrac{\underline{P}}{3\varepsilon_0}$

$\rightarrow \underline{p} = \alpha \left(\underline{E}_{diel} + \dfrac{\underline{P}}{3\varepsilon_0} \right)$

$\underline{P} = N\underline{p} = N\alpha \left(\underline{E}_{diel} + \dfrac{\underline{P}}{3\varepsilon_0} \right) \rightarrow \underline{P} \left(1 - \dfrac{N\alpha}{3\varepsilon_0} \right) = N\alpha \underline{E}_{diel}$

$\underline{P} = \chi_e \varepsilon_0 \underline{E}_{diel} \rightarrow \chi_e \varepsilon_0 \left(1 - \dfrac{N\alpha}{3\varepsilon_0} \right) = N\alpha \rightarrow \varepsilon_r = 1 + \chi_e = 1 + \dfrac{\frac{N\alpha}{\varepsilon_0}}{\left(1 - \frac{N\alpha}{3\varepsilon_0} \right)}$

Thus

$$\varepsilon_r = \frac{1 + \frac{2N\alpha}{3\varepsilon_0}}{1 - \frac{N\alpha}{3\varepsilon_0}} \rightarrow \text{Clausius - Mossotti equation.}$$

* The refractive index of a material is given by $n = \varepsilon_r^{\frac{1}{2}}$

16 Magnetic Materials

Materials whose macroscopic properties are affected by atomic magnetic dipoles

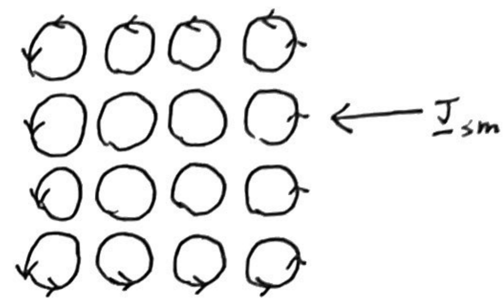
16.1 Magnetization

The atomic magnetic dipole moments are a quantum effect but we can visualize the atom as a tiny current loop.

$\underline{m}$ = atomic magnetic dipole moment = $I\underline{A}$

$\underline{A}$ = vector area enclosed by loop (right-hand-rule)

The $\underline{m}$'s align $\rightarrow$ macroscopic effect



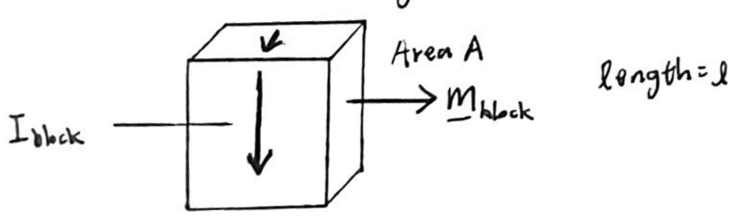
Define $\underline{M}$ = magnetization = $N\underline{m}$

= atomic magnetic dipole / volume

There is a surface current density J_{sm} in Am^{-1} due to atomic magnetic dipoles

16.2 Current densities

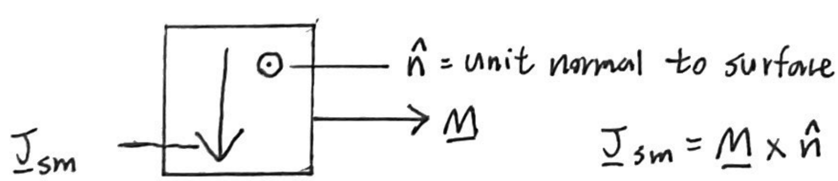
Consider a small block of magnetic material in which $\underline{M}$ ~ uniform



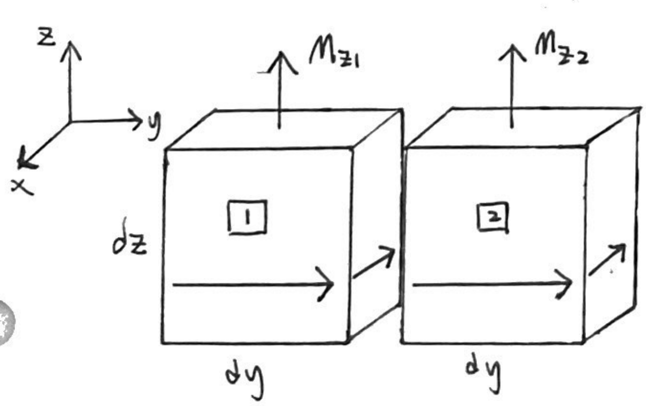
There is a 'ribbon' of current I_{block} around the surface of the block

magnetic moment of the block = $\underline{m}_{block} = \underline{M}lA$

But $m_{block} = I_{block}A \rightarrow I_{block} = Ml \rightarrow J_{sm}$ = current / width = M



$\underline{J}_{sm} = \underline{M} \times \hat{n}$



Uniform $\underline{M}$: I_{block}'s from adjacent blocks cancel

$\rightarrow$ no currents inside

Non-uniform $\underline{M}$: On mutual surface :

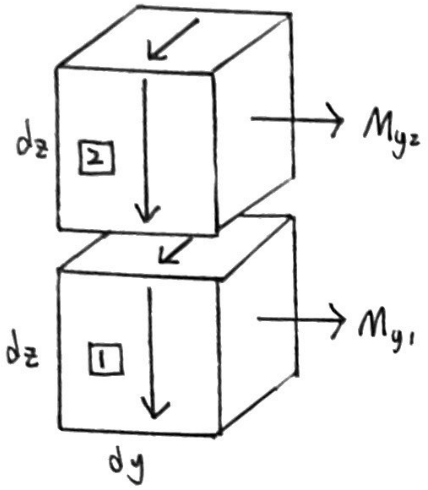
$I_x = -I_{x1} + I_{x2} = -M_{z1}dz + M_{z2}dz$

But $M_{z2} = M_{z1} + dy\dfrac{\partial M_z}{\partial y}$

$I_x = dy\,dz\,\dfrac{\partial M_z}{\partial y}$

This gives a contribution to $J_{mx} = \dfrac{\partial M_z}{\partial y}$

There is a second type of contribution to J_{mx} from M_y



On mutual surface: $I_x = I_{x1} - I_{x2} = M_{y1}dy - M_{y2}dy$

$$= -dy\,dz\frac{\partial M_y}{\partial z}$$

$\rightarrow$ Second contribution to $J_{mx} = -\dfrac{\partial M_y}{\partial z}$

No contribution from M_x (only affects J_{my} and J_{mz})

Therefore: $J_{mx} = \dfrac{\partial M_z}{\partial y} - \dfrac{\partial M_y}{\partial z} \rightarrow \underline{J}_m =$ magnetization current density $= \nabla \times \underline{M}$

In general the three types of current densities are:

(1) $\underline{J}_c, \underline{J}_{sc}$: conduction current and surface current density. (movement of free charge)

(2) $\underline{J}_m, \underline{J}_{sm}$: magnetisation current density and surface current density (bound charge)

(3) $\underline{J}_p$: polarization current density (Lecture 15)

16.3 H and μ_r

$$\nabla \times \underline{B} = \mu_0 \underline{J} + \varepsilon_0\mu_0\frac{\partial \underline{E}}{\partial t} = \mu_0\underline{J}_c + \mu_0\underline{J}_m + \mu_0\underline{J}_p + \varepsilon_0\mu_0\frac{\partial \underline{E}}{\partial t}$$

Therefore $\nabla \times \left(\dfrac{\underline{B}}{\mu_0}\right) = \underline{J}_c + \nabla \times \underline{M} + \dfrac{\partial \underline{P}}{\partial t} + \varepsilon_0\dfrac{\partial \underline{E}}{\partial t}$

$$\nabla \times \left(\frac{\underline{B}}{\mu_0} - \underline{M}\right) = \underline{J}_c + \frac{\partial}{\partial t}(\varepsilon_0\underline{E} + \underline{P})$$

Define $\underline{H} = \dfrac{\underline{B}}{\mu_0} - \underline{M}$, and recall that $\underline{D} = \varepsilon_0\underline{E} + \underline{P}$

$$\nabla \times \underline{H} = \underline{J}_c + \frac{\partial \underline{D}}{\partial t}, \quad \oint \underline{H}\cdot d\underline{l} = I_c^{encl} + \frac{d}{dt}\int \underline{D}\,d\underline{S}$$

Define $\underline{M} = \chi_m \underline{H}$, χ_m = magnetic susceptibility = scalar constant in HIL

$$\underline{H} = \frac{\underline{B}}{\mu_0} - \chi_m\underline{H} \rightarrow \underline{H} = \frac{\underline{B}}{\mu_0(1+\chi_m)} = \frac{\underline{B}}{\mu}$$

$\mu = \mu_0\mu_r$, where $\mu_r = 1 + \chi_m$ = relative permeability

16.4 Classification of magnetic materials

(1) Diamagnetic (slightly reduced applied $\underline{B}$, $\mu_r < 1$)

Apply $\underline{B} \rightarrow \frac{\partial B}{\partial t} > 0 \rightarrow$ induced $\underline{E}$

$\rightarrow$ each electron's magnetic moment is modified such that it opposes $\underline{B}_{app}$ (Lenz's law)

$\rightarrow$ $\underline{B}$ reduced

The atomic electrons have no losses

$\rightarrow$ their magnetic moments stay modified until $\underline{B}$ switched off

Diamagnetism occurs in all materials.

(2) Paramagnetic (slightly increase applied $\underline{B}$, $\mu_r > 1$)

Atoms have permanent dipole moments

- $\underline{B} = 0$: atoms are randomly oriented (thermal motion) $\rightarrow$ $\underline{M} = 0$
- $\underline{B}$ applied : $\underline{m}$'s align with $\underline{m}$ parallel to $\underline{B}$

$$\odot \underline{B}_{app} \rightarrow \odot \underline{m} \; I \rightarrow \odot \underline{B}_{dip}$$

$\rightarrow$ $\underline{B}$ increased

In a paramagnetic material this effect > diamagnetic effect

(3) Ferromagnetic (hugely increase applied $\underline{B}$)

Atoms have permanent dipole moments

- $\underline{B} = 0$: within a domain (region ~ 1mm) atomic $\underline{m}$'s are aligned, but the domains are randomly oriented.
- $\underline{B}$ applied : domains align

Ferromagnetic materials exhibit:

- non-linearity ($\underline{M}$ not proportional to $\underline{B}$), and
- hysteresis ($\underline{M}$ at any instant depends on past history, not just $\underline{B}$ at that instant)

17 Conductors

17.1 ρ_f and $\underline{J}_c$

- At some point in the conductor the number density of free electrons is N_e. Detaching an electron from an atom leaves a positively charged ion. These ions have number density N_i.
- Assuming singly charged ions, the total charge density of free charge is $\rho_f = N_i e - N_e e$
- $\underline{v}_e$ = average free electron velocity, $\underline{v}_i$ = average ion velocity
- Conduction current density (due to movement of free charges):

$$\underline{J}_c = N_i e \underline{v}_i - N_e e \underline{v}_e$$

Usually the ions are fixed in the conductor, the free electrons are mobile

$$\rightarrow \underline{J}_c = -N_e e \underline{v}_e$$

17.2 Ohm's law

$$\underline{J}_c = \sigma \underline{E} \text{ where } \sigma = \text{conductivity} = \frac{1}{\eta} \ (\eta = \text{resistivity})$$

- For a uniform current density in cylindrical wire of radius a:

$$E = \eta J \rightarrow \frac{V}{l} = \eta \frac{I}{\pi a^2} \rightarrow V = IR \text{ where } R = \frac{\eta l}{\pi a^2}$$

On the surface of a conductor we can have surface free charge density (ρ_{sf}) but assuming σ is finite we would need an infinite electric field to drive a surface conduction current in an infinitesimal surface layer, i.e. we set $\underline{J}_{sc} = 0$

17.3 Charge rearrangement time

Conservation of free charge: $\frac{\partial \rho_f}{\partial t} = -\underline{\nabla} \cdot \underline{J}_c$

In conductor $\underline{\nabla} \cdot \underline{J}_c = \sigma \underline{\nabla} \cdot \underline{E} = \frac{\sigma \rho_f}{\varepsilon} \rightarrow \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon} \rho_f$

$$\rho_{f} = \rho_{f0} e^{-t/\tau^*} \quad \text{where } \tau^* = \text{charge rearrangement time} = \frac{\varepsilon}{\sigma}$$

Apply $\underline{E}$ to conductor $\Rightarrow$ free electrons rearrange until $\underline{E} = 0$ inside.
If the only conduction currents are described by Ohm's law then $\rho_f = \rho_{f0} e^{-t/\tau^*}$
ρ_f cannot increase. If $\rho_f = 0$ initially it stays $0 \Rightarrow$ assume $\rho_f = 0$ inside a conductor, even when $\underline{E} \neq 0$

17.4 Good / poor conductors

Assume $E \propto e^{-i\omega t}$

$\underline{J}_c$ = conduction current density $= \sigma \underline{E}$

$\underline{J}_d$ = displacement current density $= \frac{\partial \underline{D}}{\partial t} = -i\omega\varepsilon \underline{E}$

$\frac{|\underline{J}_d|}{|\underline{J}_c|} = \frac{\omega\varepsilon}{\sigma} = \omega\tau^*$

Good conductor

- $\frac{\sigma}{\omega\varepsilon} >> 1$
- $\omega\tau^* << 1 \Rightarrow$ free charge rearranges itself faster than $\underline{E}$ field change.
- Free charge dominates the behaviour $\rightarrow P_p = 0$ and $\varepsilon = \varepsilon_0$
- $|\underline{J}_c| >> |\underline{J}_d|$

Poor conductor

- $\frac{\sigma}{\omega\varepsilon} << 1$
- $\omega\tau^* >> 1 \Rightarrow$ E field changes faster
- Polarization might be important $\rightarrow$ use $\varepsilon = \varepsilon_0\varepsilon_r$
- $|\underline{J}_d| >> |\underline{J}_c|$

The "good conductor" criterion depends on ω

17.5 Drude model

$E = 0$: $\underline{V}_e$ = average velocity of the free electrons $= 0$

$E \neq 0$: equation of motion for the average free electron velocity is :

$$m_e \frac{\partial V_e}{\partial t} = -e\underline{E} - \frac{m_e \underline{V}_e}{\tau_c}$$

when the second term is the rate of loss of momentum through collisions

τ_c is the electron collision time

Assume $\underline{V}_e$ and $\underline{E}$ both $\propto e^{i\omega t}$. Then the equation of motion becomes

$$-i\omega m_e \underline{V}_e = -e\underline{E} - \frac{m_e \underline{V}_e}{\tau_c} \Rightarrow \frac{1}{\tau_c}(1 - i\omega\tau_c)\underline{V}_e = -\frac{e}{m_e}\underline{E}$$

Therefore : $\underline{J}_c = -N_e e \underline{V}_e = \frac{\sigma \underline{E}}{(1 - i\omega\tau_c)}$ where $\sigma = \frac{N_e e^2 \tau_c}{m_e}$

- $\omega\tau_c << 1$: $\underline{J}_c = \sigma\underline{E}$ (collisions faster than phenomenon of interest)
- $\omega\tau_c >> 1$: $\underline{J}_c \neq \sigma\underline{E}$ (collisions can be neglected because of high frequency)

17.6 Skin effect

Good conductors: $\rho_f = 0$, $\underline{J}_d = 0$, $\varepsilon = \varepsilon_0$, and assume $\mu = \mu_0$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla} \times \left(-\frac{\partial B}{\partial t}\right)$$

$$LHS = \nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E} = -\frac{1}{\sigma} \nabla^2 \underline{J}_c.$$

$$RHS = -\frac{\partial}{\partial t}(\nabla \times \underline{B}) = -\mu_0 \frac{\partial \underline{J}_c}{\partial t} \quad (\text{since } \underline{J}_d = 0)$$

Therefore $\nabla^2 \underline{J}_c = \mu_0 \sigma \frac{\partial \underline{J}_c}{\partial t}$. This is the diffusion equation for $\underline{J}_c$

For an AC current in cylindrical wire we ~~write~~

$\underline{J}_c = j_c(r) e^{-i\omega t} \hat{z}$ where

- $j_c(r)$ is the radial profile of current density
- $e^{-i\omega t}$ is the time dependence of the current density

$$\underline{\nabla}^2 (j_c e^{-i\omega t} \hat{\underline{z}}) = \mu_0 \sigma (-i\omega j_c e^{-i\omega t} \hat{\underline{z}}) \rightarrow \nabla^2 j_c = -k^2 j_c$$

$$k^2 = i\omega\mu_0\sigma \Rightarrow k = \sqrt{\mu_0 \omega \sigma}\, i^{\frac{1}{2}} = \sqrt{\mu_0 \omega \sigma}\, \frac{1+i}{\sqrt{2}} = \frac{1+i}{\delta}, \quad \delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$$

δ is the skin depth = current diffusion length scale.

- The energy to drive the current enters the wire through the Poynting vector $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$ at the surface of the wire (radially inward)

The current penetration depth is $\sim \delta$, i.e. for a wire of radius a:

- $\underline{J}_c \sim$ uniform if $\delta >> a$, i.e. $\frac{2}{\mu_0 \sigma \omega} >> a^2 \rightarrow \omega << \frac{2}{\mu_0 \sigma a^2}$ (low f)
- $\underline{J}_c$ to a layer of width $\sim \delta$ if $\delta << a$, i.e. $\omega >> \frac{2}{\mu_0 \sigma a^2}$ (high frequencies)

$\rightarrow$ at high frequencies $R \propto \omega^{\frac{1}{2}}$

Physical Explanation

time varying I

$\rightarrow$ time varying B

$\rightarrow$ Induced E

$\rightarrow$ Induced I which opposes applied I (Lenz's law)

This effect is strongest at center $\rightarrow$ I reduced inside wire

18 Plasmas

18.1 Basics

Plasma is an ionized gas which exhibits collective behaviour

In some aspect plasmas are like ordinary gases, e.g. having a pressure

But they are also quite different:

- Ordinary gases: interactions are highly localized in space and time and only involves two particles
- Plasma: the ions and ~~ion~~ ions electrons can all act together → collective behavior.

18.2 $\underline{P}$ and $\underline{M}$

Plasmas have lots of very mobile free charges → $\underline{P} = 0$. So we don't need to use $\underline{D}$. But $\underline{M} \neq 0$

- Magnetic material: atoms have magnetic dipole moments. The atomic magnetic dipoles can be visualized as current loops. The magnetization is due to bound charges
- Plasma in magnetic field: the plasma ions and electrons follow Larmor orbits → they really are current loops. Thus there is a $\underline{M}$. Since $\underline{M}$ is due, not to bound charges, but to free charges moving in circles we can in principle calculate it. Thus we don't need $\underline{H}$.

So in plasma physics $\underline{D}$ and $\underline{H}$ are not used and $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$

In these lectures however we consider plasmas have $\underline{M} = 0$.

18.3 Current

In many plasmas the effects of collisions can be neglected.

Lecture 17: $\underline{J}_c = \frac{1}{(1 - i\omega\tau_0)} \frac{N_e e^2 \tau_c}{m_e} \underline{E}$ (where we've assumed $\underline{V}_e$ and $\underline{E} \propto e^{-i\omega t}$)

Collisionless plasma → $\omega\tau_c \gg 1$ → $\underline{J}_c = \frac{1}{-i\omega\tau_0} \frac{N_e e^2 \tau_c}{m_e} \underline{E} = \frac{N_e e^2}{m_e \omega} i \underline{E}$

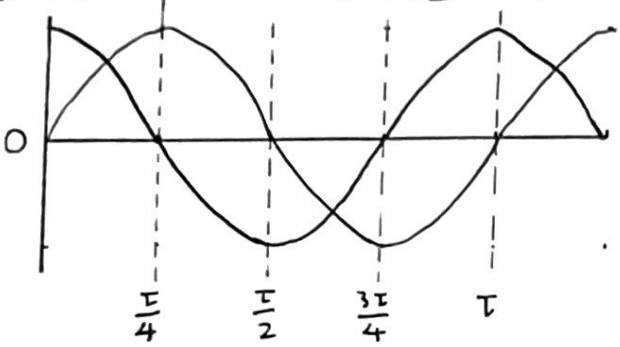
This appears to be telling us that $\underline{J}_c$ is imaginary.

$\underline{E} = \underline{E}_0 e^{-i\omega t}$. Assume the phase is such that $\underline{E}_0$ is real.

The physical field is: $E^{real} = Re(\underline{E}) = \underline{E}_0 \cos(-\omega t) = \underline{E}_0 \cos(\omega t)$

$i\underline{E} = e^{-\pi/2} \underline{E}_0 e^{-i\omega t} \rightarrow \underline{J}_c^{real} = \frac{N_e e^2}{m_e \omega} \underline{E}_0 Re\{e^{i(\pi/2 - \omega t)}\}$

But $Re\{e^{i(\pi/2-\omega t)}\} = \cos(\frac{\pi}{2} - \omega t) = \sin(\omega t)$



- $E^{real} \propto \cos(\omega t)$
- $J_c^{real} \propto \sin(\omega t)$

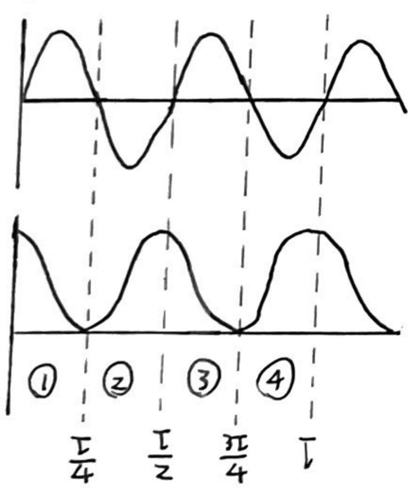
E^{real} and J_c^{real} are out of phase by $\pi/2$

18.4 Power

Power/volume into the electron is:

$\underline{J}^{real} \cdot \underline{E}^{real} = \frac{N_e e^2}{m_e \omega} E_0^2 \sin(\omega t)\cos(\omega t) = \frac{N_e e^2}{2 m_e \omega} E_0^2 \sin(2\omega t)$

Energy stored in the E field $\propto E^2 \propto \cos^2(\omega t)$



$\underline{J}_c \cdot \underline{E} \propto \sin(2\omega t)$

$E^2 \propto \cos^2(\omega t)$

1 and 3: $\underline{J}_c \cdot \underline{E} > 0 \rightarrow$ energy into electrons; field energy falls

2 and 4: $\underline{J}_c \cdot \underline{E} < 0 \rightarrow$ energy out of electrons; field energy rises

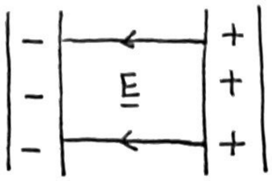
The energy goes back and forth between the field and the electrons

Time averaged over one period T: $\langle \underline{J}_c \cdot \underline{E} \rangle = \frac{1}{T}\int_0^T \frac{N_e e^2}{2 m_e \omega} E_0^2 \sin(2\omega t)\,dt = 0$

no energy loss (no collisions $\rightarrow$ no dissipation)

18.5 Plasma oscillations

Displace some electrons, leaving a positive region with excess ions $\rightarrow \rho_f \neq 0$
This charge separation sets up an electric field.



- The electrons move to the right ($m_i \gg m_e$)
- Electrons overshoot
- $\underline{E}$ reverses
- Electrons move to the left
- Overshoot ...

Ohmic conductor

Set up a charge imbalance ($\rho_f \neq 0$) somewhere

$$\rightarrow \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon} \rho_f \rightarrow \text{exponential decay of } \rho_f \quad \text{(Lecture 17)}$$

Plasma

Set up a charge imbalance ($\rho_f \neq 0$) somewhere

$$\rightarrow \frac{\partial^2 \rho_f}{\partial t^2} = -\frac{N e e^2}{m_e \varepsilon_0} \rho_f \rightarrow \text{SHM}$$

i.e. plasma oscillates at $\omega = \omega_p$ = plasma frequency $= \sqrt{\frac{N e c^2}{m_e \varepsilon_0}}$

This is an example of collective behavior.

Waves : theoretical formalism

Maxwell's equations

4-field form : $\nabla\cdot\underline{D} = \rho_f$, $\nabla\cdot\underline{B} = 0$, $\nabla\times\underline{E} = -\frac{\partial B}{\partial t}$, $\nabla\times\underline{H} = \underline{J}_c + \frac{\partial D}{\partial t}$

· ρ_p is included in $\underline{D}$

· J_m is included in $\underline{H}$

· $\underline{J}_p$ is included $\partial_t\underline{D}$

From now only consider simple materials, write Maxwell's equation in 2-field form

$\nabla\cdot\underline{E} = \frac{\rho_f}{\varepsilon}$, $\nabla\cdot\underline{B} = 0$, $\nabla\times\underline{E} = -\frac{\partial B}{\partial t}$, $\nabla\times\underline{B} = \mu\underline{J}_c + \mu\varepsilon\frac{\partial E}{\partial t}$

where $\varepsilon = \varepsilon_0\varepsilon_r$ and $\mu = \mu_0\mu_r$, assume $\mu_r = 1$

Media that will be considered

· Dielectric (HIL) : $\rho_f = 0$, $\underline{J}_c = 0$, $\varepsilon = \varepsilon_0\varepsilon_r$

· Conductor : $\rho_f = 0$, $\underline{J}_c = \sigma\underline{E}$, $\varepsilon = \varepsilon_0\varepsilon_r$

· Collisionless, unmagnetized plasma : $\rho_f \neq 0$, $\underline{J}_c = i\frac{N_e e^2}{m_e\omega}\underline{E}$, $\varepsilon = \varepsilon_0$

19.2 Plane waves

Monochromatic plane waves

$\underline{E}(\underline{r},t) = \underline{E}_0 e^{i(\underline{k}\underline{r}-\omega t)}$, $\underline{B}(\underline{r},t) = \underline{B}_0 e^{i(\underline{k}\underline{r}-\omega t)}$

The physical fields are $Re(\underline{E})$ and $Re(\underline{B})$

· $\underline{E}_0, \underline{B}_0$: amplitudes, phase, and field directions

e.g. $\underline{E}_0 = E_0\hat{e}$

E_0 is the complex amplitude of the wave, $\hat{e}$ is the unit vector in direction of $\underline{E}$

$E_0 = |E_0| e^{-i\phi_E}$, ϕ_E is the phase

E_0 and B_0 can have different phases.

Define Z = wave impedance $= \mu_0 E_0/B_0$, Z can be complex

if Z is real and positive, then E_0 and B_0 are in phase

· ω is set by source and does not change.

· $\underline{k}$ can change if the properties of the material vary

For plane wave : $\nabla \rightarrow i\underline{k}$ and $\frac{\partial}{\partial t} \rightarrow -i\omega$

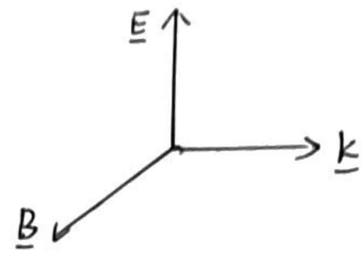
Maxwell's equation for plane waves in simple media

· $\nabla\cdot\underline{E} = \rho_f/\varepsilon \rightarrow \underline{k}\underline{E} = -i\rho_f/\varepsilon$ · $\nabla\cdot\underline{B} = 0 \rightarrow \underline{k}\cdot\underline{B} = 0$

· $\nabla\times\underline{E} = -\partial\underline{B}/\partial t \rightarrow \underline{k}\times\underline{E} = \omega\underline{B}$ · $\nabla\times\underline{B} = \mu\underline{J}_c + \mu\varepsilon\frac{\partial E}{\partial t} \rightarrow \underline{k}\times\underline{B} = -i\mu_0\underline{J}_c - \mu_0\varepsilon\omega\underline{E}$

19.3 Waves in a vacuum

In vacuum $\rho_f = J_c = 0$, $\varepsilon = \varepsilon_0$



- $\underline{k} \cdot \underline{E} = 0 \rightarrow \underline{k}$ perpendicular to $\underline{E}$
- $\underline{k} \cdot \underline{B} = 0 \rightarrow \underline{k}$ perpendicular to $\underline{B}$
- $\underline{k} \times \underline{E} = \omega \underline{B} \rightarrow \underline{B}$ perpendicular to $\underline{E}$ and $\underline{k}$
- $\underline{k} \times \underline{B} = -\mu_0 \varepsilon_0 \omega \underline{E}$

$k \times (k \times \underline{B}) = -\mu_0 \varepsilon_0 \omega \underline{k} \times \underline{E}$

LHS $= \underline{k}(\underline{k}\cdot\underline{B}) - \underline{B}(\underline{k}\cdot\underline{k})$, RHS $= -\varepsilon_0 \mu_0 \omega^2 \underline{B}$

$\rightarrow k^2 \underline{B} = \mu_0 \varepsilon_0 \omega^2 \underline{B} \rightarrow$ dispersion relation: $k^2 = \mu_0 \varepsilon_0 \omega^2$

$V_{phase} = c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$k E_0 = \omega B_0$

$Z = \frac{\mu_0 E_0}{B_0} = \frac{\mu_0 E_0}{B_0} = \frac{\mu_0 \omega}{k} = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}}$, Z is real and +ve, E_0 and B_0 in phase.

19.4 Complex vectors

The vector $\underline{k}$ can be complex i.e. $\underline{k} = \underline{k}_R + i\underline{k}_I$

where $\underline{k}_R = k_{xR}\hat{\underline{x}} + k_{yR}\hat{\underline{y}} + k_{zR}\hat{\underline{z}}$ and $\underline{k}_I = k_{xI}\hat{\underline{x}} + k_{yI}\hat{\underline{y}} + k_{zI}\hat{\underline{z}}$

In general $\underline{k}_R$ and $\underline{k}_I$ need not be parallel. Furthermore k_R and k_I tell us about different aspects of the wave behavior

There are 3 possibilities

(1) $\underline{k}$ is real:

wave propagates in $\underline{k}$, $\lambda = \frac{2\pi}{k}$, $V_{phase} = \frac{\omega}{k}$

$k = \sqrt{k^2}$ and $k^2 = \underline{k}\cdot\underline{k} = k_x^2 + k_y^2 + k_z^2 > 0$

(2) $\underline{k}$ is complex:

$$e^{i(\underline{k}\underline{r} - \omega t)} = \underbrace{e^{i(\underline{k}_R\underline{r} - \omega t)}}_{①} \underbrace{e^{i(\underline{k}_I\cdot\underline{r})}}_{②}$$

① wave propagates in $\underline{k}_R$, $k_R = \sqrt{k_{xR}^2 + k_{yR}^2 + k_{zR}^2}$

② $= e^{-\underline{k}_I\cdot\underline{r}} \rightarrow$ wave decays in the direction of $\underline{k}_I$ i.e. $\propto e^{-r/d_a}$

where d_a = attenuation distance $= 1/k_I$, $k_I = \sqrt{k_{xI}^2 + k_{yI}^2 + k_{zI}^2}$

(3) $\underline{k}$ is imaginary:

$\underline{k} = i\underline{k}_I \rightarrow e^{i(\underline{k}\underline{r} - \omega t)} = e^{-i\omega t} e^{-\underline{k}_I\underline{r}}$

This is not a propagating wave: $\underline{E}$ and $\underline{B}$ oscillate at ω and decay exponentially over attenuation distance $d_a = \frac{1}{k_I}$

20. Waves in dielectrics

20.1 Dispersion relation

Dielectric : $\rho_f = 0$, $\underline{J}_c = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$

Plane wave Maxwell's equations

- $\underline{k} \cdot \underline{E} = 0$ · $\underline{k} \times \underline{E} = \omega \underline{B}$
- $\underline{k} \cdot \underline{B} = 0$ · $\underline{k} \times \underline{B} = -\mu_0 \varepsilon \omega \underline{E}$

Assuming $\underline{k}$ is real, $\underline{k}, \underline{E}, \underline{B}$ are mutually perpendicular vectors.

$\underline{k} \times (\underline{k} \times \underline{B}) = -\mu_0 \varepsilon \omega \, \underline{k} \times \underline{E}$

$LHS = \underline{k} \cdot (\underline{k} \cdot \underline{B}) - \underline{B}(\underline{k} \cdot \underline{k})$, $RHS = -\varepsilon \mu_0 \omega^2 \underline{B}$

$\rightarrow k^2 \underline{B} = \mu_0 \varepsilon \omega^2 \underline{B} \rightarrow$ dispersion relation : $k^2 = \mu_0 \varepsilon \omega^2$

$V_{phase} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}}$ $[\varepsilon_r > 1 \rightarrow V_{phase} < c]$

Define n = refractive index $= \frac{c}{V_{phase}} = \varepsilon_r^{1/2}$

Dispersion relation can be written : $k = \frac{\omega n}{c}$

$k E_0 = \omega B_0$

$Z = \frac{\mu_0 E_0}{B_0} = \frac{\mu_0 \omega}{k} = \frac{\mu_0}{\sqrt{\mu_0 \varepsilon}} = \frac{1}{n}\sqrt{\frac{\mu_0}{\varepsilon_0}}$

If n is real and positive then E_0 and B_0 are in phase.

20.2 Oscillator model

Apply $\underline{E}$ $\rightarrow$ outer shell electrons displaced from nucleus $\rightarrow$ atomic dipole moment

$\underline{E} = -E\hat{\underline{x}}$ ⊕ ⊖

$x\hat{\underline{x}}$ electron displacement from equilibrium

$\underline{p} = q\underline{d}$, $q = |$total charge$|$ on electrons in atom's outer shell

$\underline{d} = -x\hat{\underline{x}}$ (from $-$ to $+$)

Treat the atom as an oscillator undergoing forced oscillations

$\underline{F}_1$ = force from $\underline{E} = -q(-E\hat{\underline{x}}) = qE\hat{\underline{x}}$

$\underline{F}_2$ = restoring force $= -Kx\hat{\underline{x}}$

where K is spring constant $m\omega_0^2$ (m is m_e and ω_0 is oscillator natural frequency)

$\underline{F}_3$ = damping force $= -\underline{v} m \gamma$

where $\underline{v} = \frac{d}{dt}(x\hat{\underline{x}})$ and γ = damping constant (real and positive)

Therefore $m\frac{d^2}{dt^2}(x\hat{\underline{x}}) = \underline{F}_1 + \underline{F}_2 + \underline{F}_3 = qE\hat{\underline{x}} - m\omega_0^2 x\hat{\underline{x}} - m\gamma\frac{d}{dt}(x\hat{\underline{x}})$

$$\rightarrow \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{q}{m}E$$

We are considering forced oscillations at ω = frequency of wave, i.e. $\frac{d}{dt} \rightarrow -i\omega$, $\frac{d^2}{dt^2} \rightarrow -\omega^2$

$$\rightarrow (-\omega^2 - i\omega\gamma + \omega_0^2)x = \frac{q}{m}E$$

$$\underline{p} = q\underline{d} = -qx\hat{\underline{x}} = -q\hat{\underline{x}}\left\{\frac{-\frac{q}{m}E}{\omega^2 + i\omega\gamma - \omega_0^2}\right\} = \frac{-q^2\underline{E}}{m(\omega^2 - \omega_0^2 + i\omega\gamma)} \quad (\text{since } \underline{E} = -E\hat{\underline{x}})$$

Therefore: $\underline{D} = \varepsilon_0\underline{E} + N\underline{p} = \varepsilon_0\underline{E} - \frac{Nq^2}{m(\omega^2 - \omega_0^2 + i\omega\gamma)}\underline{E}$

$$\rightarrow \varepsilon_r = 1 - \frac{Nq^2}{\varepsilon_0 m(\omega^2 - \omega_0^2 + i\omega\gamma)}$$

20.3 Dispersion

Usually we can neglect $i\omega\gamma \rightarrow \varepsilon_r = 1 - \frac{Nq^2}{\varepsilon_0 m(\omega^2 - \omega_0^2)}$

ε_r increases with ω

$n = \sqrt{\varepsilon_r}$ increases with ω $\rightarrow$ V_{phase} ($\frac{c}{n}$) decreases with ω

A medium in which the phase velocity of a wave depends on its frequency is said to be dispersive.

For visible light: $\omega_{blue} \approx 4 \times 10^{15}\ s^{-1}$, $\omega_{red} \approx 2.7 \times 10^{15}\ s^{-1}$

$\rightarrow n_{blue} > n_{red}$ $\rightarrow$ blue light travels more slowly through a dielectric.

20.4 Anomalous dispersion

If the driving frequency of the wave is close to the natural frequency of the atomic oscillator ($\omega \sim \omega_0$) the damping term $i\omega\gamma$ is important.

Damping $\rightarrow$ wave energy is dissipated (absorbed by dielectric)

Keeping the $i\omega\gamma$ term will make ε_r, n, and k complex

Assume wave propagates in $\hat{\underline{z}}$, $\underline{k} = (k_R + ik_I)\hat{\underline{z}}$

Then $e^{i(\underline{k}\cdot\underline{r} - \omega t)} = e^{i(k_R z - \omega t)}e^{-k_I z}$

This represents a wave with wavelength $\lambda = \frac{2\pi}{k_R}$ and amplitude $\propto e^{-k_I z}$

Since the wave is attenuated the dielectric is ~ opaque at these frequencies

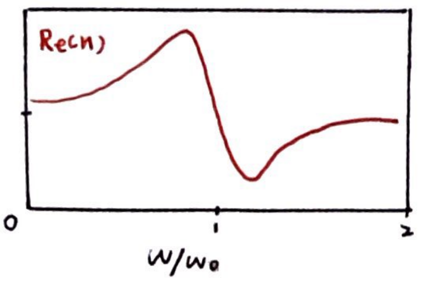
For low density matter:

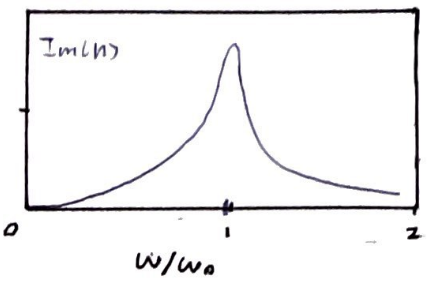
$$n = \sqrt{\varepsilon_r} = \left\{1 - \frac{Nq^2}{\varepsilon_0 m}\frac{1}{(\omega^2 - \omega_0^2 + i\omega\gamma)}\right\}^{1/2} \simeq 1 - \frac{Nq^2}{2\varepsilon_0 m}\frac{1}{\omega^2 - \omega_0^2 + i\omega\gamma}$$

where we have used binomial expansion.

Writing $\omega_d^2 = \frac{Nq^2}{2\varepsilon_0 m}$ we have

$$Re(n) = 1 - \frac{\omega_d^2(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}, \quad Im(n) = \frac{\omega_d^2\omega\gamma}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}$$





$$V_{phase} = \lambda f = \frac{\omega}{k_R} = \frac{c}{Re(n)}$$

Close to $\omega = \omega_0$: $Re(n)$ decreases with ω $\rightarrow$ V_{phase} increases with ω $\rightarrow$ anomalous dispersion.

21 Waves in plasmas

21.1 Basic equations

Plasma (collisionless, unmagnetized) : ρ_f need not be zero, $\underline{J}_c = i\frac{N_e e^2}{m_e \omega}\underline{E}$, $\varepsilon = \varepsilon_0$

Plane wave Maxwell's equations

- $\underline{k}\cdot\underline{E} = -i\frac{\rho_f}{\varepsilon_0}$ · $\underline{k}\cdot\underline{B} = 0$
- $\underline{k}\times\underline{E} = \omega\underline{B}$ · $\underline{k}\times\underline{B} = -i\mu_0\underline{J}_c - \varepsilon_0\mu_0\omega\underline{E}$

$$-i\mu_0\underline{J}_c = \frac{\mu_0 N_e e^2}{m_e\omega}\underline{E} \rightarrow \underline{k}\times\underline{B} = \varepsilon_0\mu_0\omega\left(\frac{N_e e^2}{m_e\varepsilon_0\omega^2} - 1\right)\underline{E} = \frac{\omega}{c^2}\left(\frac{\omega_p^2}{\omega^2} - 1\right)\underline{E}$$

where ω_p = plasma frequency $= \sqrt{\frac{N_e e^2}{m_e\varepsilon_0}}$ (if N_e given in m^{-3} then $\omega_p \approx 56.4\sqrt{N_e}\,s^{-1}$)

21.2 Longitudinal solution

$\underline{B} = 0$ (not electromagnetic)

- $\underline{k}\times\underline{E} = 0$ i.e. $\underline{E}$ parallel to $\underline{k} \rightarrow \rho_f = i\varepsilon_0\underline{k}\cdot\underline{E} \neq 0$
- $\underline{k}\times\underline{B} = 0 \rightarrow \frac{\omega_p^2}{\omega^2} - 1 = 0 \rightarrow \omega_p = \omega$, ω independent of k $\rightarrow$ no propagation

- This is a longitudinal electrostatic oscillation in which a charge separation ($\rho_f \neq 0$) produces an $\underline{E}$ field which provides the restoring force, i.e. plasma oscillation
- $\tau_p = 2\pi/\omega_p$ is the timescale on which electrons can respond to E fields in the plasma. This is analogous to τ^*, charge rearrangement time in ohmic conductors.

21.3 Transverse solution

$\underline{B} \neq 0$ (electromagnetic)

- $\underline{k}\cdot(\underline{k}\times\underline{B}) = \frac{\omega}{c^2}\left(\frac{\omega_p^2}{\omega^2} - 1\right)\underline{k}\cdot\underline{E}$, LHS $= 0 =$ RHS $\rightarrow \underline{k}\cdot\underline{E} = 0$

  $\underline{E}$ is perpendicular to $\underline{k}$ (i.e. transverse), and $\rho_f = 0$
- $\underline{k}\times(\underline{k}\times\underline{B}) = \frac{\omega}{c^2}\left(\frac{\omega_p^2}{\omega^2} - 1\right)\underline{k}\times\underline{E}$

  LHS $= -k^2\underline{B}$ since $\underline{k}\cdot\underline{B} = 0$, RHS $= \frac{\omega}{c^2}\left(\frac{\omega_p^2}{\omega^2} - 1\right)\omega\underline{B} = \left(\frac{\omega_p^2 - \omega^2}{c^2}\right)\underline{B}$

  Therefore $\left[k^2 + \frac{(\omega_p^2 - \omega^2)}{c^2}\right]\underline{B} = 0$

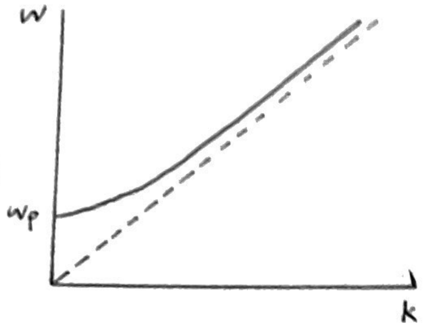
  Dispersion relation : $k^2 = \frac{\omega^2 - \omega_p^2}{c^2} \rightarrow k = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}$

$$Z = \frac{\mu_0 E_0}{B_0} = \frac{\mu_0\omega}{k} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-\frac{1}{2}}\sqrt{\frac{\mu_0}{\varepsilon_0}}$$

21.3.1 High frequency $\omega > \omega_p$

$k^2 > 0 \rightarrow k$ is real

The wave $\propto e^{i(\underline{k}\cdot\underline{r} - \omega t)}$ propagates in the direction of $\underline{k}$.



Large k: $\omega/k \approx c$

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \rightarrow \frac{V_{phase}}{c} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-\frac{1}{2}}$$

- V_{phase} depends on ω → plasmas are dispersive
- $V_{phase} > c$, But information travels at group velocity $V_{group} = \frac{d\omega}{dk}$

$$k^2c^2 = \omega^2 - \omega_p^2 \rightarrow 2c^2k\,dk = 2\omega\,d\omega \rightarrow \frac{d\omega}{dk} = \frac{c^2}{\omega/k} = \frac{c^2}{V_{phase}} < c$$

21.3.2 Low frequency $\omega < \omega_p$

$k^2 < 0 \rightarrow k$ is imaginary

Write $\underline{k} = i\underline{k}_I$, then $e^{i(\underline{k}\cdot\underline{r} - \omega t)} = e^{-\underline{k}_I\cdot\underline{r}}\,e^{-i\omega t}$

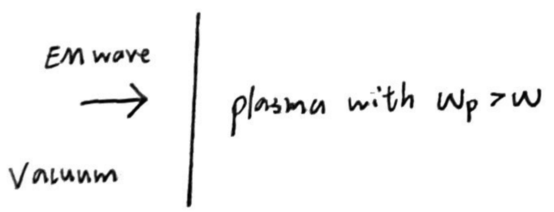
This is an oscillation with angular frequency ω which falls exponentially over attenuation distance $d_a = 1/k_I$, i.e. no propagation for $\omega < \omega_p$

$\omega < \omega_p \rightarrow T$ = period of wave $> T_p$

The plasma electrons have time to respond to the wave E field.

21.4 Critical Density

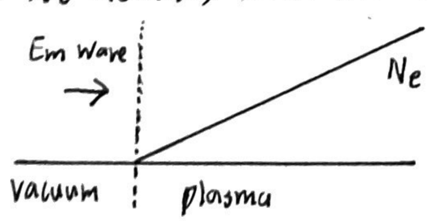
Consider an em wave with given ω, encountering a plasma from a vacuum



In plasma:

- Oscillating $\underline{E}$ and $\underline{B}$ due to wave
- no energy dissipated i.e. wave is reflected

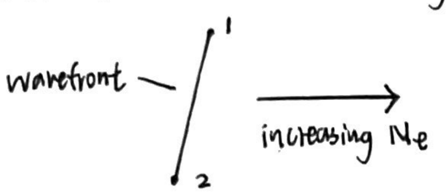
Suppose the plasma has a density ramp
i.e. N_e increases with distance → ω_p increases with distance

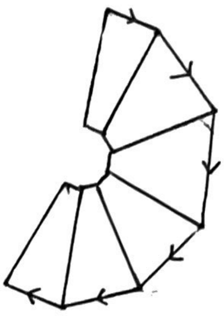


The wave cannot penetrate beyond the surface on which $\omega_p = \omega$

→ the wave is reflected when $\left(\frac{N_e e^2}{m_e \varepsilon_0}\right)^{1/2} = \omega$

i.e. at the critical density: $N_e^{crit} = \frac{m_e \varepsilon_0 \omega^2}{e^2}$





N_e increases → ω_p increases

→ $V_{phase} = \frac{c}{\sqrt{1-\omega_p^2/\omega^2}}$ increases

In time Δt point 1 travels further than 2

→ the wave is reflected

The ionosphere is a low density plasma with a peak electron number density $N_e \sim 10^{12}\ m^{-3}$ at height of ~ 250 km

This corresponds to a frequency $f_p = \frac{\omega_p}{2\pi} \simeq 10^7$ Hz

Radio waves with $f < 10$ MHz are reflected from the ionosphere, enabling transcontinental communication at these frequencies.

22 Waves in conductors

22.1 Displacement relation

Ohmic conductor : $\rho_f = 0$, $\underline{J}_c = \sigma \underline{E}$, $\epsilon = \epsilon_0 \epsilon_r$

Plane wave Maxwell's equations :

- $\underline{k} \cdot \underline{E} = 0 \rightarrow \underline{k} \perp \underline{E}$
- $\underline{k} \cdot \underline{B} = 0 \rightarrow \underline{k} \perp \underline{B}$
- $\underline{k} \times \underline{E} = \omega \underline{B}$
- $\underline{k} \times \underline{B} = -i\mu_0 \underline{J}_c - \epsilon \mu_0 \omega \underline{E}$

$$\underline{k} \times (\underline{k} \times \underline{B}) = -i\mu_0 \underline{k} \times \underline{J}_c - \mu_0 \epsilon \omega \underline{k} \times \underline{E}$$

$$LHS = -k^2 \underline{B}$$

$$RHS = -i\mu_0 \sigma \underline{k} \times \underline{E} - \mu_0 \epsilon \omega \underline{k} \times \underline{E} = -\mu_0 \epsilon \omega \left(\frac{i\sigma}{\epsilon\omega} + 1\right) \omega \underline{B}$$

$\rightarrow$ dispersion relation : $k^2 = \mu_0 \epsilon \omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right)$

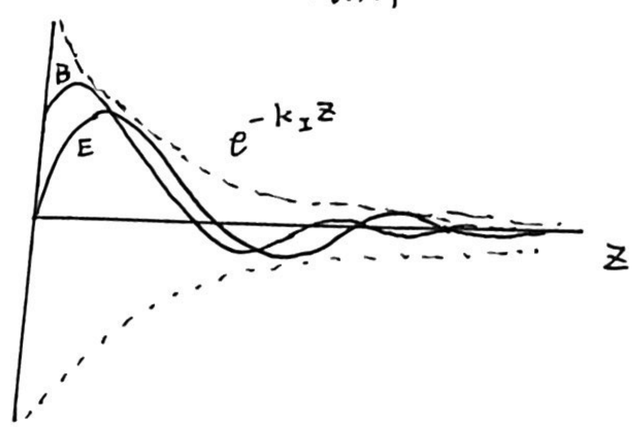
$\rightarrow$ k is complex

- Wave propagates in $\underline{k}_R$ with $\lambda = 2\pi / k_R$ and $V_{phase} = \omega / k_R$
- Wave decays in the direction of $\underline{k}_I$ with $d_a = 1/k_I$

Here we assume $\underline{k}_R$ and $\underline{k}_I$ are in the same direction

$$k_R + i k_I = |k| e^{i\phi} = |k|(\cos\phi + i \sin\phi)$$

$$\rightarrow \tan\phi = k_I / k_R$$



$$\underline{k} \cdot \underline{E} = \omega \underline{B} \rightarrow k E_0 = \omega B_0$$

$$E_0 = |E_0| e^{i\phi_E} , \quad B_0 = |B_0| e^{i\phi_B}$$

$$\rightarrow |k| e^{i\phi} |E_0| e^{i\phi_E} = \omega |B_0| e^{i\phi_B}$$

$\rightarrow \phi_B = \phi_E + \phi$ i.e. $\underline{B}$ lags $\underline{E}$ by ϕ

$$Z = \frac{\mu_0 E_0}{B_0} = \frac{\mu_0 \omega}{k} = \frac{\mu_0 \omega}{k_R + i k_I}$$

22.2 Poor conductor

Lecture 17: $\frac{\omega \epsilon}{\sigma} >> 1$

$$k^2 = \mu_0 \epsilon \omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right) \rightarrow k = \omega \sqrt{\mu_0 \epsilon} \left(1 + \frac{i\sigma}{\epsilon\omega}\right)^{\frac{1}{2}} \simeq \omega \sqrt{\mu_0 \epsilon} \left(1 + \frac{i\sigma}{2\epsilon\omega}\right)$$

$$\rightarrow k_R = \omega \sqrt{\mu_0 \epsilon} , \quad k_I = \frac{\sigma}{2} \sqrt{\frac{\mu_0}{\epsilon}}$$

$$V_{phase} = \frac{\omega}{k_R} = \frac{1}{\sqrt{\mu_0 \epsilon}} = \frac{1}{\sqrt{\epsilon_r}} \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{c}{\sqrt{\epsilon_r}} , \quad \epsilon_r \geqslant 1 \rightarrow V_{phase} \leqslant c$$

$$\frac{k_I}{k_R} = \frac{\sigma}{2} \sqrt{\frac{\mu_0}{\epsilon}} \frac{1}{\omega \sqrt{\mu_0 \epsilon}} = \frac{\sigma}{2\epsilon\omega} << 1$$

$\frac{d_a}{\lambda} = \frac{1}{k_I} \frac{k_R}{2\pi} >> 1 \rightarrow$ slight attenuation

$\tan\phi = k_I / k_R \rightarrow \phi$ is small $\rightarrow E_0$ and B_0 almost in phase

Define refractive index using the dielectric dispersion relation

$n = \frac{ck}{\omega} = \frac{c}{\omega}(k_R + ik_I) \rightarrow$ complex n with $k_I/k_R \ll 1$

A poor conductor is basically a dielectric which conducts very slightly. slight dissipation (Joule heating) $\rightarrow$ slight attenuation of the wave.

22.3 Good conductor

Lecture 17: $\varepsilon \rightarrow \varepsilon_0$ and $\frac{\omega\varepsilon_0}{\sigma} \ll 1$

$k^2 = \mu_0 \varepsilon \omega^2 (1 + \frac{i\sigma}{\varepsilon\omega}) \simeq i\mu_0\sigma\omega$

$\underline{k} = \underline{k}_R + i\underline{k}_I \rightarrow k^2 = (\underline{k}_R + i\underline{k}_I)(\underline{k}_R + i\underline{k}_I) = k_R^2 - k_I^2 + 2i\underline{k}_R \cdot \underline{k}_I$

- $\mathrm{Re}(k^2) = 0 \rightarrow k_R^2 = k_I^2 \rightarrow k_R = k_I$
- $\mathrm{Im}(k^2) = \mu_0 \sigma \omega$

Assume $\underline{k}_R$ and $\underline{k}_I$ are parallel $\rightarrow 2\underline{k}_R \cdot \underline{k}_I = 2k_R k_I = 2k_R^2 = \mu_0\sigma\omega$

i.e. $k_R = k_I = \frac{1}{\delta}$

where $\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$, skin depth

$V_{phase} = \frac{\omega}{k_R} = \omega\delta = \omega\sqrt{\frac{2}{\mu_0\sigma\omega}} = \sqrt{\frac{2\omega}{\mu_0\sigma}} = \sqrt{\frac{2\omega\varepsilon_0}{\sigma}}\frac{1}{\sqrt{\varepsilon_0\mu_0}}$

i.e. $V_{phase} = \sqrt{\frac{2\omega\varepsilon_0}{\sigma}}\, c \ll c$

$\lambda = \frac{2\pi}{k_R} = 2\pi\delta$, $d_a = \frac{1}{k_I} = \delta = \frac{\lambda}{2\pi} \rightarrow$ strong attenuation

$\tan\phi = \frac{k_I}{k_R} = 1 \rightarrow B_0$ lags E_0 by $\frac{\pi}{4}$.

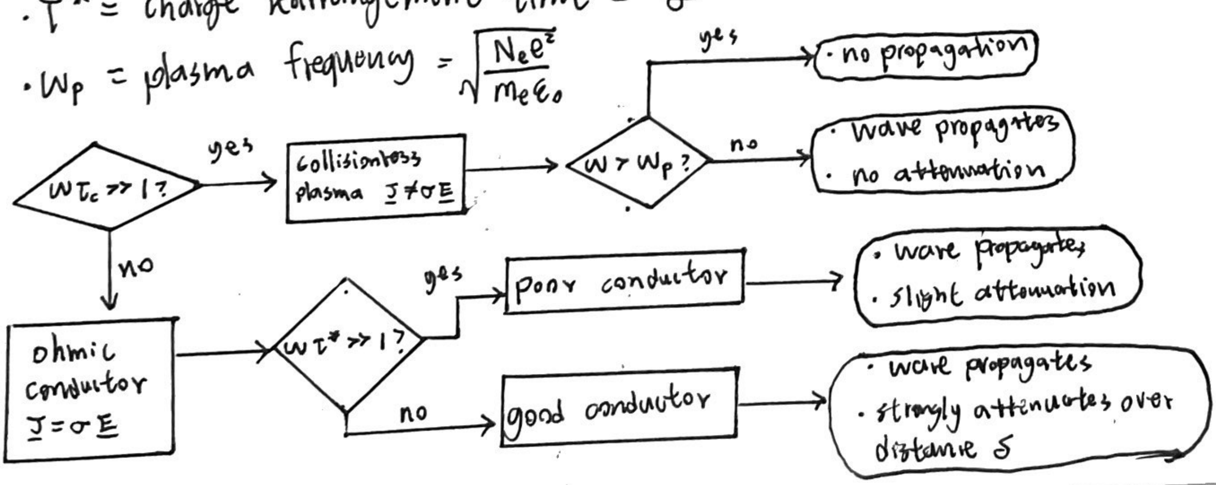
$$Z = \frac{\mu_0 \omega}{k_R + ik_I} = \frac{\mu_0\omega\delta}{1+i}$$

22.4 Are metals conductors or plasmas?

It depends on ω

We also need the values of the following parameters

- τ_c = electron collision time $= \frac{m_e \sigma}{N_e e^2}$
- τ^* = charge rearrangement time $= \frac{\varepsilon_0}{\sigma}$
- ω_p = plasma frequency $= \sqrt{\frac{N_e e^2}{m_e \varepsilon_0}}$

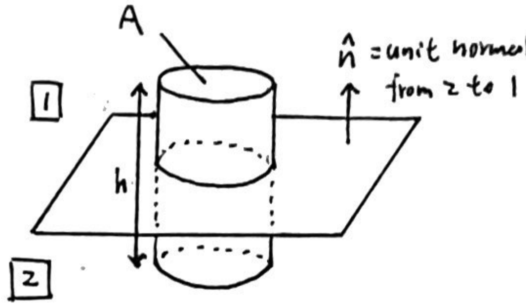


23 Boundaries : theoretical formalism

23.1 Boundary conditions

We need b.c.s which apply to electromagnetic fields at an interface
On the interface there can be surface free charge (ρ_{sf}) and surface conduction current ($\underline{J}_{sc}$)

(1) $\oint \underline{D} \cdot d\underline{s} = Q_f^{encl}$



The integral over the closed pill box:

$$\oint \underline{D} \cdot d\underline{s} = \int_{top} \underline{D} \cdot d\underline{s} + \int_{base} \underline{D} \cdot d\underline{s} + \int_{side} \underline{D} \cdot d\underline{s}$$

$$\int_{top} \underline{D} \cdot d\underline{s} = (\hat{n} \cdot \underline{D}_1) A$$

$$\int_{base} \underline{D} \cdot d\underline{s} = -(\hat{n} \cdot \underline{D}_2) A$$

We take the limit $h \to 0$, $\lim_{h \to 0} \int_{side} \underline{D} \cdot d\underline{s} \to 0$

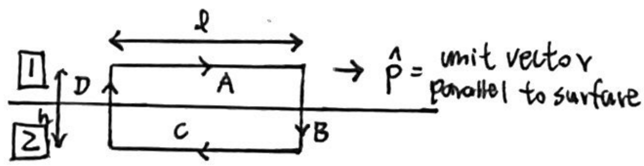
$$\lim_{h \to 0} Q_f^{encl} \to \rho_{sf} A$$

Therefore: $(\hat{\underline{n}} \cdot \underline{D}_1 - \hat{\underline{n}} \cdot \underline{D}_2) A = \rho_{sf} A \to D_{1\perp} - D_{2\perp} = \rho_{sf}$
where $\perp$ denotes perpendicular from medium 2 to medium 1.

(2) $\oint \underline{B} \cdot d\underline{s} = 0$

Using the same sort of Gaussian pillbox gives $B_{1\perp} - B_{2\perp} = 0$

(3) $\oint \underline{E} \cdot d\underline{l} = -\frac{d}{dt} \int \underline{B} \cdot d\underline{s}$



The integral round the loop can be split:

$$\oint \underline{E} \cdot d\underline{l} = \int_A \underline{E} \cdot d\underline{l} + \int_B \underline{E} \cdot d\underline{l} + \int_C \underline{E} \cdot d\underline{l} + \int_D \underline{E} \cdot d\underline{l}$$

$$\int_A \underline{E} \cdot d\underline{l} = (\hat{p} \cdot \underline{E}_1) l, \quad \int_C \underline{E} \cdot d\underline{l} = -(\hat{p} \cdot \underline{E}_2) l$$

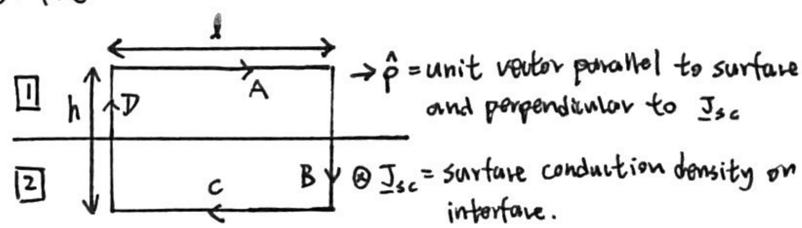
We take the limit $h \to 0$

$$\lim_{h \to 0} \int_B \underline{E} \cdot d\underline{l} \to 0 \text{ and } \lim_{h \to 0} \int_D \underline{E} \cdot d\underline{l} \to 0$$

$\lim_{h \to 0} \int \underline{B} \cdot d\underline{s} \to 0$ (since $\underline{B}$ is everywhere finite and area $\to$ zero)

Therefore $(\hat{p} \cdot \underline{E}_1 - \hat{p} \cdot \underline{E}_2) l = 0 \quad \to \quad E_{1\parallel} - E_{2\parallel} = 0$
where $\parallel$ denotes parallel to surface

(4) $\oint \underline{H} \cdot d\underline{l} = I_c^{encl} + \frac{d}{dt} \int \underline{D} \cdot d\underline{s}$



In the limit $h \to 0$

- $\oint \underline{H} \cdot d\underline{l} \to \hat{p} \underline{H}_1 l - \hat{p} \cdot H_2 l$
- $I_c^{encl} = J_{sc} l$
- $\int \underline{D} \cdot d\underline{S} \to 0$ (since $\underline{D}$ is everywhere finite and area $\to$ zero)

Therefore: $(\hat{p} \underline{H}_1 - \hat{p} \cdot \underline{H}_2) l = J_{sc} l \to H_{1\parallel} - H_{2\parallel} = J_{sc}$

For simple materials with $\mu_r = 1$

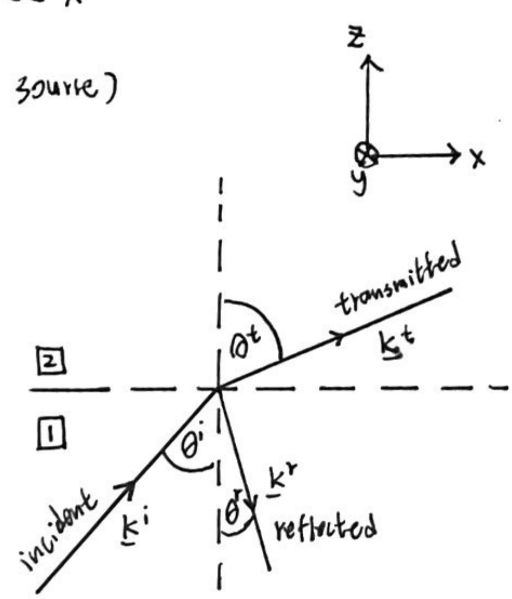
- $\varepsilon_{r1} E_{1\perp} - \varepsilon_{r2} E_{2\perp} = \rho_{sf} / \varepsilon_0$
- $E_{1\parallel} - E_{2\parallel} = 0$
- $B_{1\perp} - B_{2\perp} = 0$
- $B_{1\parallel} - B_{2\parallel} = \mu_0 J_{sc}$

23.2 Basic Setup

A plane electromagnetic wave in medium 1 encounters an interface with medium 2. Some of the incident wave's energy is reflected back into medium 1, and some goes into medium 2.

Assume:

- The radius of curvature of the interface $\gg \lambda \to$ we can consider a small flat section.
- The thickness of the transition region from 1 to 2 $\ll \lambda$
  $\to$ we consider the interface as a discontinuity
- The three waves have the same ω (set by the source)
- Each wave has its own $\underline{k}$, $\underline{E}_0$, and $\underline{B}_0$.



Notation and jargon

- Subscripts: $\perp, \parallel$ to the interface
- Plane of incidence: plane defined by $\hat{n}$ = unit normal to interface, and $\underline{k}^i$ (i.e. the plane of the paper in the figure)

Define the coordinates such that the interface is the x-y plane and the plane of incidence is the x-z plane. We assume $\underline{k}^i$ is real.

23.3 Basic Properties

The electric field in the two media are

$$\underline{E}_1 = \underline{E}_0^i e^{i(\underline{k}^i \cdot \underline{r} - \omega t)} + \underline{E}_0^r e^{i(\underline{k}^r \cdot \underline{r} - \omega t)}, \quad \underline{E}_2 = \underline{E}_0^t e^{i(\underline{k}^t \cdot \underline{r} - \omega t)}$$

Coordinates defined such that $z=0$ everywhere on the interface and $k_y^i = 0$

Continuity of $E_\parallel$ at the interface:

$$E_{0\parallel}^i e^{i(k_x^i x - \omega t)} + E_{0\parallel}^r e^{i(k_x^r x + k_y^r y - \omega t)} - E_{0\parallel}^t e^{i(k_x^t x + k_y^t y - \omega t)} = 0$$

$$\to E_{0\parallel}^i + E_{0\parallel}^r e^{i(k_x^r - k_x^i) x} e^{i k_y^r y} - E_{0\parallel}^t e^{i(k_x^t - k_x^i) x} e^{i k_y^t y} = 0$$

The 1st term is independent of x and y → the other terms must be too

- $k^r_y = k^t_y = 0$, the reflected and transmitted waves are in the plane of incidence.
- $k^r_x = k^t_x = k^i_x$

For the given ω we can solve the dispersion relations in the two media to find k_1 = magnitude of $\underline{k}$ in medium 1 and k_2 = magnitude of $\underline{k}$ in medium 2.

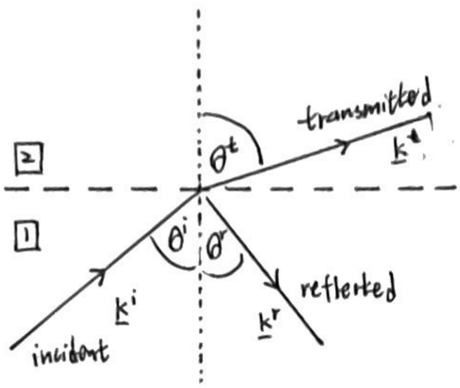
In medium 1: $k^i_x = k_1 \sin\theta^i$, $k^r_x = k_1 \sin\theta^r$

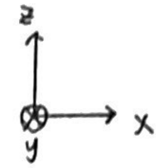
$k^r_x = k^i_x \Rightarrow \sin\theta^r = \sin\theta^i \Rightarrow \theta^r = \theta^i$

i.e. the angle of reflection = the angle of incidence.

24. Snell's law and the Fresnel equations

24.1 Snell's law





Continuity of $E_{\parallel}$ at the interface $\rightarrow k^t_x = k^i_x$

Normal incidence : $\theta^i = 0 \rightarrow k^i_x = 0 \rightarrow k^t_x = 0 \rightarrow \theta^t = 0$

For given ω we can solve the dispersion relations in the two media to find k_1 = magnitude of $\underline{k}$ in medium 1 and k_2 = magnitude of $\underline{k}$ in medium 2.

Assume both media are HIL dielectrics, with refractive indices n_1 and n_2. Dielectric dispersion relation can be written : $k = \frac{n\omega}{c}$

$\rightarrow k^i_x = k_1 \sin\theta^i = \frac{n_1\omega}{c}\sin\theta^i$ and $k^t_x = k^t \sin\theta^t = \frac{n_2\omega}{c}\sin\theta^t$

$\rightarrow n_1 \sin\theta^i = n_2 \sin\theta^t$ (Snell's law)

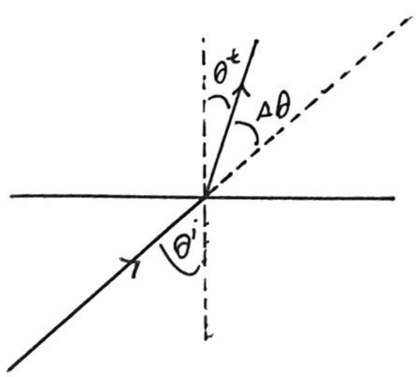
- $n_1 < n_2$ (low n to high n, e.g. air to glass) : $\theta^t < \theta^i$
- $n_1 > n_2$ (high n to low n, e.g. glass to air) : $\theta^t > \theta^i$

If $n_1 > n_2$ and $\sin\theta^i > n_2/n_1$ then Snell's law shows $\sin\theta^t > 1$

i.e. Snell's law breaks down if $\theta^i > \theta_c$, $\theta_c = \sin^{-1}(n_2/n_1)$

24.2 Prisms

- Air to glass



$\Delta\theta$ = deviation angle $\theta^i - \theta^t$

$n_{glass} \approx 1.5$ is assumed

But glass is dispersive (Lecture 20) : v_{phase} decreases with ω.

Snell's law : $\sin\theta^t = \frac{1}{n_{glass}}\sin\theta^i$

Increase n for given $\theta^i \rightarrow \theta^t$ decreases $\rightarrow \Delta\theta$ increases

i.e. higher frequencies have a larger deviation angle

$\omega_{blue} > \omega_{red} \rightarrow$ prism deflects blue light more than red.

24.3 Fresnel equations

Equations for:

t = amplitude transmission coefficient $= E_0^t / E_0^i$,
r = amplitude reflection coefficient $= E_0^r / E_0^i$,
for arbitrary angle of incidence.

r and t tells us how the field amplitude and phases change at the surface.

$E_0^i = |E_0^i| e^{i\phi^i}$, $E_0^r = |E_0^r| e^{i\phi^r}$, $E_0^t = |E_0^t| e^{i\phi^t}$

$r = \frac{|E_0^r|}{|E_0^i|} e^{i(\phi^r - \phi^i)} \rightarrow |r| = |E_0^r| / |E_0^i|$, $\arg(r) = \phi^r - \phi^i$ = phase change on reflection

Similarly $|t| = |E_0^t| / |E_0^i|$, $\arg(t) = \phi^t - \phi^i$ = phase change on transmission.

Assume medium 1 is a dielectric and medium 2 either another dielectric or an ohmic conductor. For both dielectrics and conductors:

- $B_0 = \mu_0 E_0 / Z$, where Z is the wave impedance of the medium.
- $J_{sc} = 0$

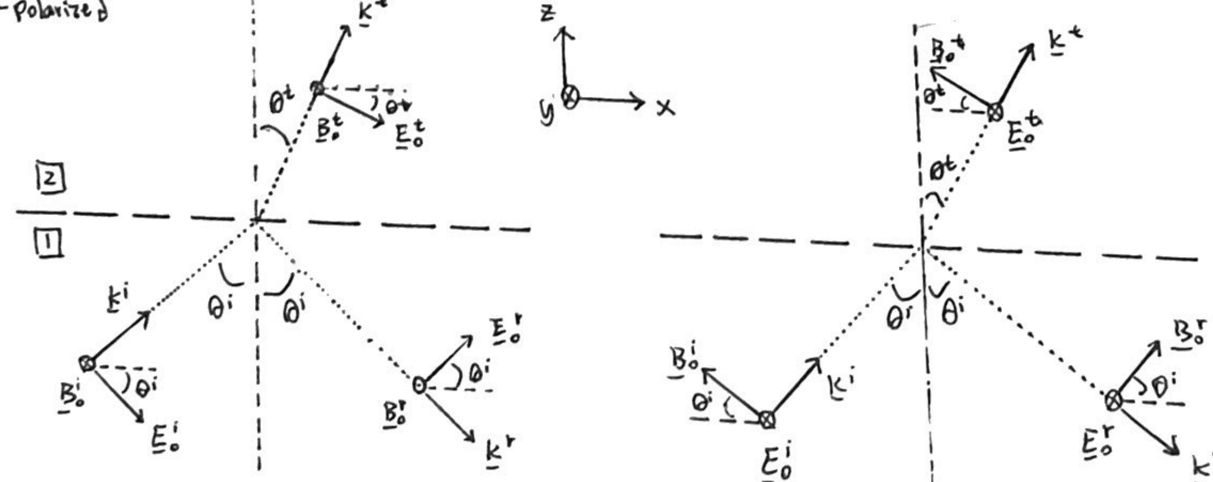
Boundary conditions used

- $E_{0\parallel}$ is continuous across interface
- $B_{0\parallel}$ is continuous across interface

It is important to consider the polarisation of the incident wave (i.e. the direction of $\underline{E}_0^i$). An incident wave of any arbitrary polarization can be formed by a superposition of two linearly polarized waves:

- p-polarized: $\underline{E}^i$ parallel to plane of incidence ($\rightarrow \underline{B}^i$ parallel to interface)
- s-polarized: $\underline{E}^i$ perpendicular to plane of incidence ($\rightarrow \underline{E}^i$ parallel to interface)

P-polarized



For both polarisations: $B_0^i = \frac{\mu_0 E_0^i}{Z_1}$, $B_0^r = \frac{\mu_0 E_0^r}{Z_1}$, $B_0^t = \frac{\mu_0 E_0^t}{Z_2}$

24.3.1 p-polarization

The boundary conditions are :

- $E^i_{0x} + E^r_{0x} = E^t_{0x} \rightarrow E^i_0 \cos\theta^i + E^r_0 \cos\theta^i = E^t_0 \cos\theta^t$ (P1)
- $B^i_0 - B^r_0 = B^t_0 \rightarrow \dfrac{E^i_0}{Z_1} - \dfrac{E^r_0}{Z_1} = \dfrac{E^t_0}{Z_2}$ (P2)

These equations give

$$t_p = \frac{2\cos\theta^i}{\cos\theta^t + (Z_1/Z_2)\cos\theta^i}, \quad r_p = \frac{\cos\theta^t - (Z_1/Z_2)\cos\theta^i}{\cos\theta^t + (Z_1/Z_2)\cos\theta^i}$$

24.3.2 s-polarized

The boundary conditions are :

- $E^i_0 + E^r_0 = E^t_0$ (S1)
- $-B^i_0 \cos\theta^i + B^r_0 \cos\theta^i = -B^t_0 \cos\theta^t \rightarrow -\dfrac{E^i_0}{Z_1}\cos\theta^i + \dfrac{E^r_0}{Z_1}\cos\theta^i = -\dfrac{E^t_0}{Z_2}\cos\theta^t$

These equations give (S2)

$$t_s = \frac{2\cos\theta^i}{\cos\theta^i + (Z_1/Z_2)\cos\theta^t}, \quad r_s = \frac{\cos\theta^i - (Z_1/Z_2)\cos\theta^t}{\cos\theta^i + (Z_1/Z_2)\cos\theta^t}$$

25 Boundaries between various media I

25.1 Media 1 and 2 both dielectrics

Dielectric : $Z = \frac{1}{n}\sqrt{\frac{\mu_0}{\varepsilon_0}} \rightarrow \frac{Z_1}{Z_2} = \frac{n_2}{n_1}$

We assume n_1 and n_2 are real (i.e. neglect anomalous dispersion)

$$t_p = \frac{2\cos\theta^i}{\cos\theta^t + (n_2/n_1)\cos\theta^i}, \quad r_p = \frac{\cos\theta^t - (n_2/n_1)\cos\theta^i}{\cos\theta^t + (n_2/n_1)\cos\theta^i}$$

$$t_s = \frac{2\cos\theta^i}{\cos\theta_i + (n_2/n_1)\cos\theta^t}, \quad r_s = \frac{\cos\theta^i - (n_2/n_1)\cos\theta^t}{\cos\theta^i + (n_2/n_1)\cos\theta^t}$$

Note :

- We need Snell's law to find θ^t for given θ^i
- At normal incidence we have $r_p = r_s$ and $t_p = t_s$
- t_p and t_s are positive $\rightarrow$ E_0^t and E_0^i are in phase
- r_p and r_s can be negative. If so $\rightarrow$ E_0^r and E_0^i have a phase diff of π.
- $r_p = 0$ at $\theta^i = \theta_B$ = Brewer angle = $\tan^{-1}\left(\frac{n_2}{n_1}\right)$

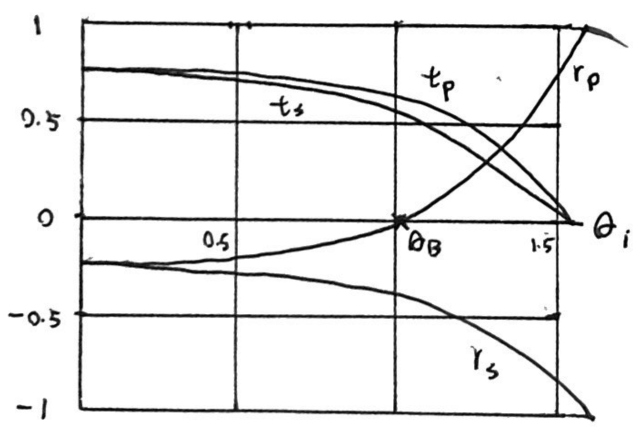
25.1.1 Low n to high n

$t_p > 0,\ t_s > 0 \rightarrow \phi^t - \phi^i = 0$

$r_s < 0 \rightarrow$ for s-polarized : $\phi^r - \phi^i = \pi$

$r_p \begin{cases} < 0 \text{ if } \theta^i < \theta_B \\ > 0 \text{ if } \theta^i > \theta_B \end{cases} \rightarrow$ for p-polarized : $\phi^r - \phi^i \begin{cases} = \pi \text{ if } \theta^i < \theta_B \\ = 0 \text{ if } \theta^i > \theta_B \end{cases}$

E.g. air to glass ($n_1 = 1$, $n_2 = 1.5$)



25.1.2 High n to low n

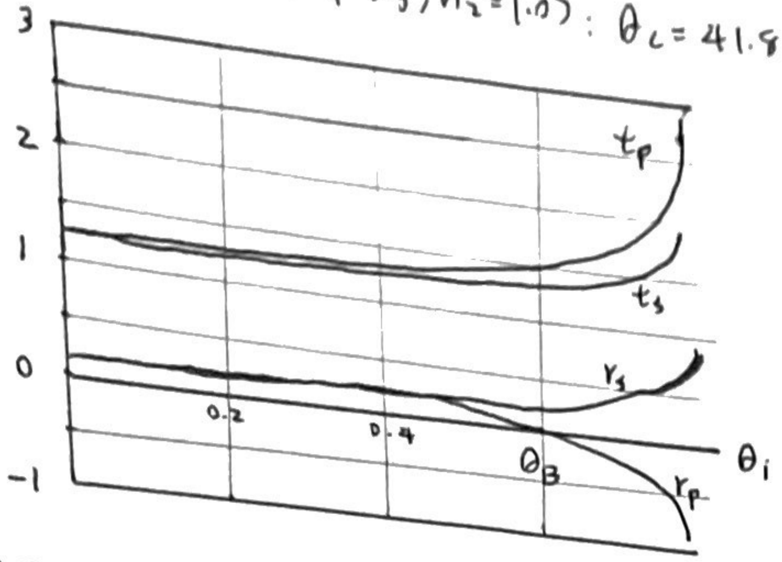
Only consider $0 \le \theta^i \le \theta_c$, where θ_c = critical angle

$t_p > 0,\ t_s > 0 \rightarrow \phi^t - \phi^i = 0$

$r_s > 0 \rightarrow$ for s-polarized : $\phi^r - \phi^i = 0$

$r_p \begin{cases} > 0 \text{ if } \theta^i < \theta_B \\ < 0 \text{ if } \theta^i > \theta_B \end{cases} \rightarrow$ for p-polarized : $\phi^r - \phi^i \begin{cases} = 0 \text{ if } \theta^i < \theta_B \\ = \pi \text{ if } \theta^i > \theta_B \end{cases}$

E.g. glass to air ($n_1 = 1.5, n_2 = 1.0$): $\theta_c = 41.8°$ (0.73 rad)



25.2 Medium 1 = vacuum, medium 2 = good conductor, normal incidence.

25.2.1 Reflection from a good conductor

Normal incidence: $\theta^i = 0 \rightarrow \theta^t = 0$

$$\rightarrow r_p = r_s = r = \frac{1 - Z_1/Z_2}{1 + Z_1/Z_2}$$

Vacuum: $Z = \sqrt{\frac{\mu_0}{\epsilon_0}}$

Good conductor: $Z = \frac{\mu_0 \omega \delta}{1+i}$, where $\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$

$$\frac{Z_1}{Z_2} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{(1+i)}{\mu_0 \omega \delta} = (1+i)\sqrt{\frac{1}{\epsilon_0 \mu_0 \omega^2} \frac{\mu_0 \sigma \omega}{2}} = \frac{1+i}{\zeta}, \text{ where } \zeta = \sqrt{\frac{2\epsilon_0 \omega}{\sigma}} \ll 1$$

Therefore $r = \frac{1-(1+i)/\zeta}{1+(1+i)/\zeta} = \frac{\zeta-(1+i)}{\zeta+(1+i)} \approx -1$

$\arg(r)$ = phase ~~phase~~ change on reflection $\approx \pi$

R = fraction of wave energy reflected from surface $= \frac{|E_0^r|^2}{|E_0^i|^2} = |r|^2$

But $r = \frac{(\zeta-1)-i}{(\zeta+1)+i} \rightarrow R = \frac{(\zeta-1)^2+1}{(\zeta+1)^2+1} = \frac{\zeta^2-2\zeta+2}{\zeta^2+2\zeta+2} \approx \frac{-2\zeta+2}{2\zeta+2}$ ($\zeta \ll 1$)

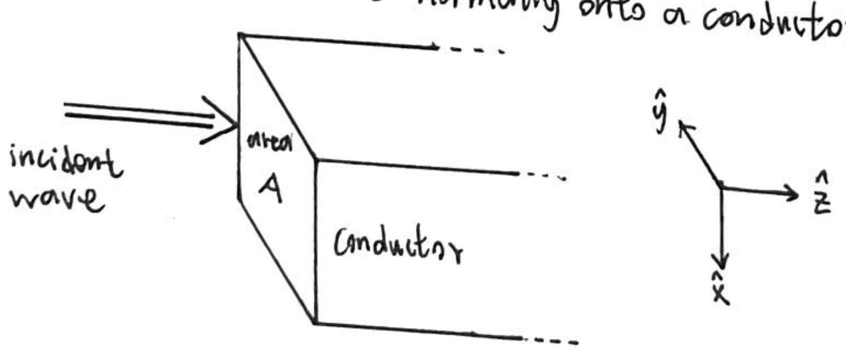
$\rightarrow R = \frac{1-\zeta}{1+\zeta} \approx (1-\zeta)(1-\zeta)$ [using binomial expansion] $\approx 1-2\zeta$

i.e. $R \approx 1 - \sqrt{\frac{8\epsilon_0 \omega}{\sigma}}$ (Hagen-Rubens relation)

$R \approx 1 \rightarrow$ nearly all wave energy reflected from a good conductor.

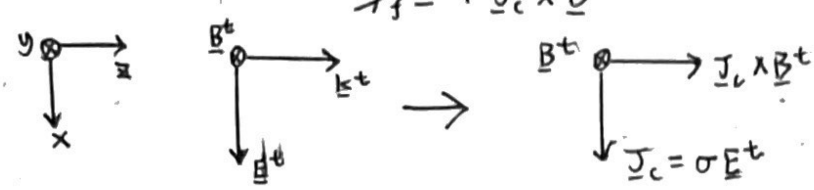
25.2.2 Radiation pressure

Consider an EM wave incident normally onto a conductor



E^t of wave drives a conduction current ($\underline{J}_c = \sigma \underline{E}^t$) in conductor. This current experiences a force due to $\underline{B}^t$ of wave.

Force/volume in conductor $= N_i\{e\underline{E} + (e\underline{v}_i) \times \underline{B}\} + N_e\{-e\underline{E} + (-e\underline{v}_e) \times \underline{B}\}$
$= (N_i - N_e)e\underline{E} - N_e e \underline{v}_e \times \underline{B}$
$= \cancel{\rho_f \underline{E}} + \underline{J}_c \times \underline{B}$



Forces in $+z$ direction, i.e. as if incident wave exerts a pressure on conductor.

Good conductor: $J_c \gg J_d \rightarrow \underline{J}_c = \frac{1}{\mu_0} \nabla \times \underline{B}^t = -\frac{1}{\mu_0} \frac{\partial B^t}{\partial z} \hat{x}$

Force/volume $= \underline{J}_c \times \underline{B}^t = -\frac{1}{\mu_0} B^t \frac{\partial B^t}{\partial z} \hat{z} = -\frac{\partial}{\partial z}\left(\frac{B^{t2}}{2\mu_0}\right)\hat{z}$

$\underline{F} = |\text{Force}| = \int_{Vol} (\underline{J}_c \times \underline{B}^t) = -\hat{z}A \int_0^\infty \frac{\partial}{\partial z}\left(\frac{B^{t2}}{2\mu_0}\right) dz = -\hat{z}A\left[\frac{B^{t2}}{2\mu_0}\right]_{z=0}^{z=\infty}$

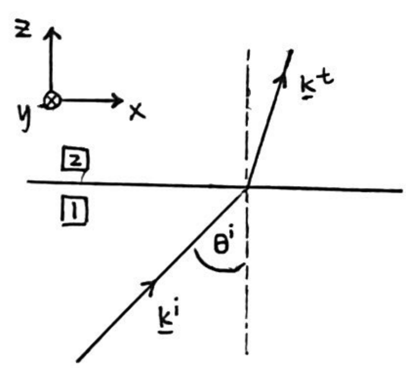
But $B^t \propto e^{-z/\delta}$ (lecture 22) $\rightarrow B^t \rightarrow 0$ as $z \rightarrow \infty$ i.e. $\underline{F} = +\hat{z}A\left(\frac{B^{t2}}{2\mu_0}\right)_{z=0}$

P_{rad} = radiation pressure $= \left\langle \frac{|\underline{F}|}{A} \right\rangle \approx \frac{2}{c}\langle S^i \rangle$.

where $\langle S^i \rangle$ is the intensity of incident wave.

26 Boundaries between various media II

26.1 $\underline{k}^t$



$\underline{k}^t$ can be complex, i.e. $\underline{k}^t = \underline{k}^t_R + i\underline{k}^t_I$

- $k^t_y = 0$
- $k^t_x = k^i_x$ = real

$\rightarrow \underline{k}^t = k^t_x \hat{x} + (k^t_{zR} + i k^t_{zI})\hat{z}$

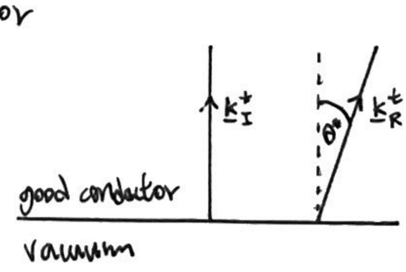
$\rightarrow \underline{k}^t_R = k^t_x \hat{x} + k^t_{zR}\hat{z}$ and $\underline{k}^t_I = k^t_{zI}\hat{z}$

- direction of $\underline{k}_R$ = direction of wave propagation
- direction of $\underline{k}_I$ = direction in which wave amplitude decays

26.2 Medium 1 = vacuum, medium 2 = good conductor

In medium 2 $\underline{k}^t_I$ is normal to the surface but $\underline{k}^t_R$ need not be.

Define θ^* = angle between normal to interface and $\underline{k}^t_R$, i.e. angle at which wave propagates



$k^{t2} = (\underline{k}_R^t + i\underline{k}_I^t)(\underline{k}_R^t + i\underline{k}_I^t) = k_R^{t2} - k_I^{t2} + 2i\underline{k}_R^t \underline{k}_I^t$

But $k^{t2} = k^2 = i\mu_0\sigma\omega$ (lecture 22)

Equate real and imaginary parts

- Real: $k_R^{t2} - k_I^{t2} = 0 \rightarrow k_R^t = k_I^t$
- Imaginary: $2\underline{k}_R^t \underline{k}_I^t = \mu_0\sigma\omega$

LHS $= 2k_{zR}^t k_I^t$ (since $\underline{k}_I^t$ is in z-direction)

$= 2k_R^t \cos\theta^* k_R^t$ (since $k_{zR}^t = k_R^t\cos\theta^*$ and $k_R^t = k_I^t$)

$\rightarrow k_R^{t2}\cos\theta^* = \dfrac{\mu_0\sigma\omega}{2}$ $\quad\left(\dfrac{\mu_0\sigma\omega}{2} = \dfrac{1}{\delta^2},\ \delta \text{ is skin depth}\right)$ (I)

$k_x^t = k_x^i$

LHS $= k_{xR}^t = k_R^t \sin\theta^*$

RHS $= k_1 \sin\theta^i = \dfrac{\omega}{c}\sin\theta^i$

$\rightarrow k_R^{t2}\sin^2\theta^* = \omega^2\varepsilon_0\mu_0\sin^2\theta^i$ (II)

(II)/(I): $\dfrac{\sin^2\theta^*}{\cos\theta^*} = \dfrac{2\omega\varepsilon_0}{\sigma}\sin^2\theta^i << 1$ i.e. θ^* is very small.

This implies:

(i) wave in a good conductor propagates ~ normal to the surface for arbitrary angle of incidence.

(ii) both $\underline{k}_R^t$ and $\underline{k}_I^t$ are ~ in z-direction

Assumption that $\underline{k}_R$ and $\underline{k}_I$ are parallel is justified.

26.3 Media 1 and 2 both dielectrics

Dielectrics: $k = \dfrac{n\omega}{c}$ (Vacuum is dielectric with $n=1$)

$k_x^t = \dfrac{n_1\omega}{c}\sin\theta^i \rightarrow \underline{k}_t = \dfrac{n_1\omega}{c}\sin\theta^i\hat{x} + k_z^t\hat{z}$

But $\underline{k}^t \cdot \underline{k}^t = k_2^2 \rightarrow \dfrac{n_1^2\omega^2}{c^2}\sin^2\theta_i + k_z^{t2} = \dfrac{n_2^2\omega^2}{c^2}$, $\left(k_z^{t2} = \dfrac{\omega^2}{c^2}(n_2^2 - n_1^2\sin^2\theta^i)\right)$

If Snell's law is applicable: $n_1\sin\theta^i = n_2\sin\theta^t$

$\rightarrow k_z^{t2} = \dfrac{n_2^2\omega^2}{c^2}(1-\sin^2\theta^t) \rightarrow k_z^t = k_2\cos\theta^t$

But Snell's law fails if $n_1 > n_2$ and $\theta^i > \theta_c$

$\rightarrow \sin\theta^i > \dfrac{n_2}{n_1} \rightarrow k_z^{t2} < 0$

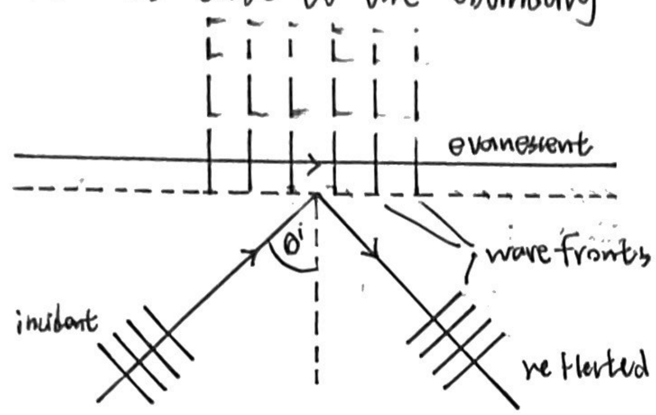
In this case k_x^t $(= k_1\sin\theta^i)$ is real, but k_z^t is imaginary

26.4 Evanescent waves

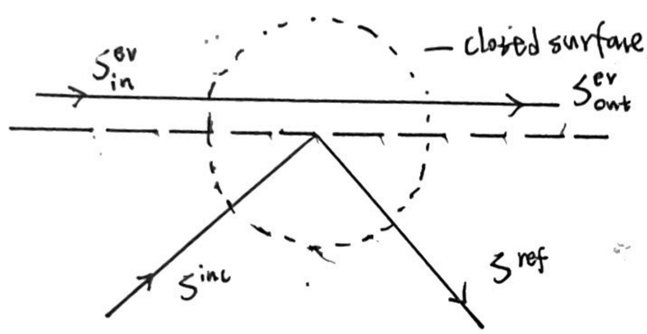
In medium 2 the wave

- propagates in the x direction with $\lambda = 2\pi / k_x^t$ and
- decays in the z direction (perpendicular to boundary) with $d_a = 1/k_z^t$

i.e. wave is localized close to the boundary



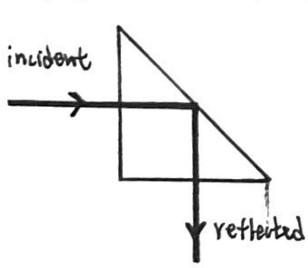
All 3 waves carry energy. Consider a closed surface enclosing part of the interface

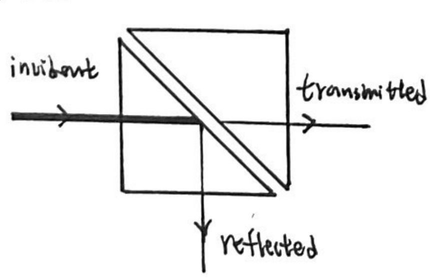


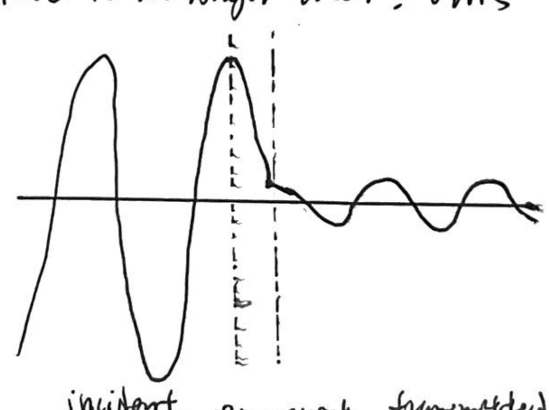
The total energy flow through the surface must be zero

i.e. energy flow in incident wave = energy flow in reflected wave → TIR

A beam of light incident onto a glass prism undergoes total internal reflection. Now bring a 2nd prism close to the first one. There is now a 2nd interface (vacuum → glass). at which the evanescent wave couples to a transmitted wave in the glass. This transimitted wave carries away some of the incident wave energy. Thus the internal reflection at the first interface is no longer total. This is called frustrated TIR.







This is exactly analogous to quantum tunnelling.