Investigating the Amplitude Dependence of a Pendulum's Period

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***Abstract*—The amplitude dependence of a pendulum's period is investigated by calculating the period of a simple gravity pendulum at different angular displacements. Videos of pendulum motions are fed to python codes that calculate the periods. A graph of period against amplitude is plotted and a quadratic curve is fitted, which has coefficients that are close to zero (5.3068×10^{-4} and -1.557229×10^{-5} with absolute uncertainties of 2×10^{-8} and 5×10^{-11} respectively.), showing that the pendulum's period, as well the the period in simple harmonic motion, is amplitude independent.**

I. Introduction

Simple gravity pendulums are often used to describe simple harmonic motions. The first scientific investigation of pendulums was by Galileo Galilei in 1602[1]. A pendulum is a weight suspended on a string from a pivot. When a pendulum is moved sideways from its original position, or equilibrium position, a restoring force due to gravity will drive the bob towards the equilibrium position, causing the pendulum to swing back and forth[2]. The regular motion of a pendulum made it popular for timekeeping, as the motion of a pendulum can be approximated to a simple harmonic motion at small angular displacements. In this experiment, the effect of amplitude, or angular displacement, change on period will be investigated by using a simple gravity pendulum. The experiment is significant in ways that the result validates the dynamics of simple harmonic motion which are crucial for research in different fields, such as solar oscillation, where oscillations of the sun and its nearby stars are measured to gain understanding of their internal structure[3].

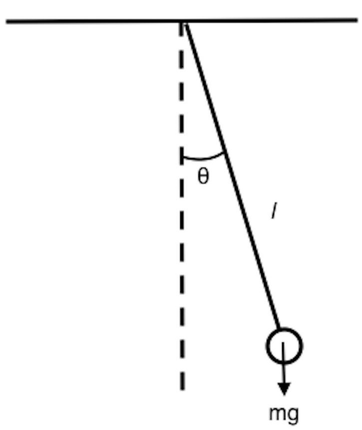


Fig. 1: A Simple Gravity Pendulum with a length l and an amplitude θ

II. Theory

Figure 1 shows the setup of a simple gravity pendulum with length l and angle θ from the vertical. Using Newton's second law, we can describe the motion of the bob with mass m by[4]:

$$-ml\ddot{\theta} = mgsin\theta \tag{1}$$

By using the small angle approximation $sin\theta \approx \theta$, the pendulum approximates a simple harmonic motion. The period of an pendulum in simple harmonic motion is dependent only by its length and the value of g; the equation is given by:

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{2}$$

Where T is the period of the pendulum. In this experiment, python codes that are fed with videos of the pendulum motion captured by the camera are used to analyze the motion of the pendulum; the codes calculate the period of the motion by fitting the data points with a sine curve. However, since the motion of a simple gravity pendulum is not exactly simple harmonic, since small angle approximation is used, the period calculated by the python codes will have increased uncertainties as amplitude increases even within the small angle range. Therefore it is necessary to consider the true period of the simple gravity pendulum without the premise that its motion is simple harmonic. The true period of an ideal simple gravity pendulum is given by the Legendre polynomial solution for elliptic integral[5]:

$$T = 2\pi\sqrt{\frac{l}{g}}\left[\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 sin^{2n}\left(\frac{\theta_0}{2}\right)\right] \tag{3}$$

The difference between equations (2) and (3) will account for the increased uncertainties for the value of period as amplitude increases. A plot of period against amplitude is then plotted using the same python codes. Simplifying equation (3) by assuming that the period of a finite amplitude pendulum can be expressed as a correction to the small amplitude period gives[4]:

$$T = T_0(1 + \alpha\theta_0 + \beta\theta_0^2) \tag{4}$$

Where T_0 is the small amplitude period, α and β are constants to be determined using python, and θ_0 is the amplitude of the oscillation. The equation for period hence becomes a quadratic function and if α and β are close to zero, the period of a pendulum will be amplitude independent.

III. Method

Figure 2 shows the setup of the experiment. For the python code to perform best the background should create a sharp contrast to the pendulum, hence a dark board is used. A thread of around 30 cm and a brass bob are used for this experiment. In this experiment the length of the pendulum is fixed(28.7cm). Before filming starts, the camera will be turned on to check if the camera view is unobstructed and if the entire trajectory of the pendulum can be captured. The swinging of the pendulum is made as horizontally as possible so it has a small movement in the y direction in the camera view. For each amplitude, 5 oscillations will be recorded. Amplitude is measured by fixing a protractor on the pendulum clamp so that when the pendulum is at equilibrium position, the string and the 0° line superpose. The period will be measured at 4 amplitudes: 5°, 10°, 15°, and 20°. Since small angle approximation is used to derive equation (2), the maximum amplitude for this experiment will be 20°. After feeding each video to the python code, two separate plots will be displayed, showing the fitted sine function ($asin(bx + c) + d$, where a, b, c, and d are constants, see appendix for that section of code) based on the motion of the pendulum in the x and y direction against time. The period will also be calculated and is given by $\frac{2\pi}{b}$, as well as its associated uncertainties.

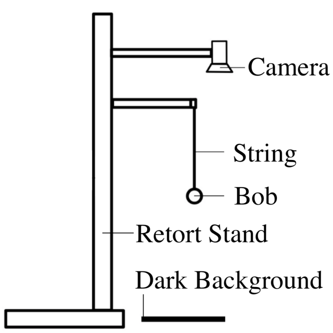
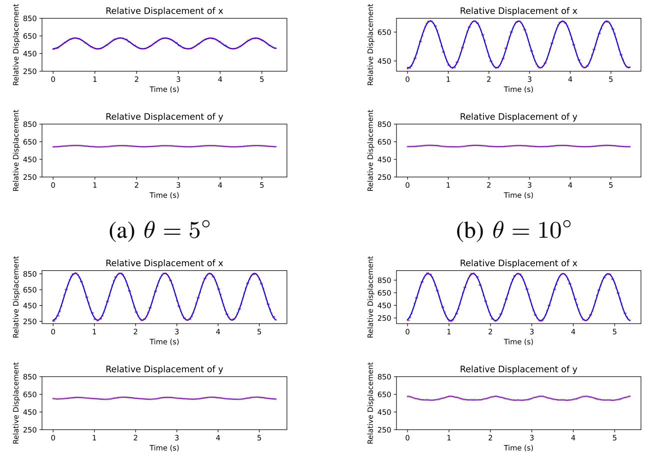


Fig. 2: Experimental Setup – the camera used to record the video is above the pendulum and parallel to the dark background

IV. Results

The graphs in figure three show the movements of the pendulum at different amplitudes in both the x and the y direction. The amplitude of the sine waves in the y direction for all amplitudes are small compared to those in the x direction. This indicates small movements of the pendulum in the y direction. The sine curves fit the points well as all the crosses are very close to the curve in all of the graphs. The amplitude of the sine curves in the x direction reflect the amplitude at which the bob is being released. As the amplitude increases, the relative wave height in the x direction also increases. Moreover, for all curves corresponding the x direction, there are exactly 5 oscillations, which are consistent with the video input. The time period can be determined by measuring the horizontal distance between two consecutive



(a) $\theta = 5°$ (b) $\theta = 10°$ (c) $\theta = 15°$ (d) $\theta = 20°$

Fig. 3: Motions of pendulum in x and y directions for different amplitudes, plotted with python

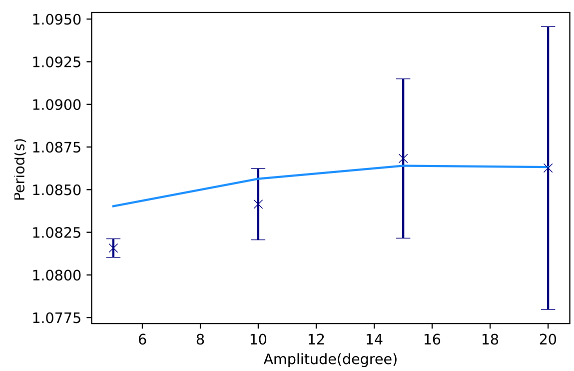


Fig. 4: The graph of period against amplitude, plotted with python

peaks of the sine wave, but the more accurate way is to find it using the python code which also gives it's uncertainty. Figure 4 shows a graph of period against amplitude. The values of period(see appendix) are from the python code and the values of uncertainties are from the python code and equation (3). The values of α and β after curve fitting equation (4) are 5.3068×10^{-4} and -1.557229×10^{-5} with absolute uncertainties of $2{\times}10^{-8}$ and $5{\times}10^{-11}$ respectively. The values of α and β are not significantly different from zero and the curve showed relatively small absolute gradient throughout the graph. Both equation (2) and figure 4 suggest that the period of a simple gravity pendulum is independent of amplitude. In this experiment, the pendulum is approximated as a simple harmonic oscillator by using small angle approximation, so the experiment also suggests that in simple harmonic motion, period is independent of amplitude. The factors that could have affected the result of this experiment include measurements of the amplitude and the length of the string used for the pendulum, since the calculation of the uncertainty of period depend on equation (3). The value of g can also be a potential source of uncertainty that might have affected the result, since the earth's gravitational field is not constant throughout its surface. The uncertainty induced by the camera is small and insignificant compared to the uncertainties that preceded.

V. Conclusion

The experiment aims to investigate the amplitude dependence of a simple gravity pendulum's period. Small angles are used so that the pendulum's motion can be approximated as simple harmonic motion. The Legendre polynomial solution for elliptic integral is used to find uncertainties of the period due to the fact that the simple gravity pendulum motion is not perfectly simple harmonic. A camera is used to record all the oscillations and the 30 fps videos are fed to python codes that calculate the period. A graph of period against amplitude is plotted with a quadratic curve fit where α and β are not significantly different from 0, showing that the pendulum's period, as well the the period in simple harmonic motion, is not amplitude dependent. This investigation showed me new insights in the underlying significance of the small angle approximation and the systematic ways of determining one variable's dependence of another.

References

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[2] Halliday, D., Resnick, R. and Walker, J., 2020. Fundamentals of physics. Milton: John Wiley, pp.423-429.

[3] University of Birmingham. 2021. Why is simple harmonic motion so important?. [online] Available at: ¡https://www.birmingham.ac.uk/teachers/study-resources/stem/physics/harmonic-motion.aspx¿ [Accessed 7 December 2021].

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Appendix

The python code for the sine curve fitting of the motion of the pendulum in the x and y directions

```
timee = frame/30
def test_func(x, a, b,c,d):
    return a * np.sin(b*x+c)+d
params, params_covariance = optimize.curve
_fit(test_func, timee, yy, p0=[300,6,3/4,500])
def test_func2(x, a, b,c,d):
    return a * np.sin(b*x+c)+d
params2, params_covariance2 = optimize.cu
rve_fit(test_func2, timee, xx, p0=[10,6,0
.1, 600])
```

TABLE I: Table of Period and Absolute Uncertainty for Amplitude

Amplitude (degree)	Period (s)	Total absolute uncertainty (s)
20	1.086	8×10^{-3}
15	1.087	5×10^{-3}
10	1.084	2×10^{-3}
5	1.0816	5×10^{-4}